SREE VAHINI INSTITUTE OF SCIENCE AND TECHNOLOGY
(AN AUTONOMOUS INSTITUTION)
An ISO 9001 : 2015 Certified Institution
Diploma / B.Tech / M.Tech / BBA & BCA / MBA
Sai Vahini Nagar, NH-30, Tiruvuru-NTR Dist. A.P
College Code : VHNI

DEPARTMENT OF MATHEMATICS

STUDY MATERIAL

IB.TECH

LINEAR ALGEBRA AND CALCULUS

(SUB CODE : V23111CC22)

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FOREWORD

It is our distinct privilege to present this compendium of Linear Algebra and Calculus specially prepared for first-year B.Tech students under the I-Scheme curriculum. Mathematics lies at the heart of all engineering disciplines; this text aims to provide a rigorous yet accessible foundation in key topics that underpin advanced courses in sciences and technology.

This volume has been structured to blend formal definitions, illustrative examples, and a diverse set of exercises to foster both conceptual understanding and problem-solving skills. The choice of topics, their organization, and the pedagogical progression have been guided by the overarching objective of aligning with the prescribed syllabus while addressing the needs of a varied student cohort.

We extend our profound gratitude to **Dr. R. Nagendra Babu, Principal**, for endorsing and supporting this endeavour; and to **Dr. K. Kiran Kumar, Dean (Academics)**, for the academic oversight, alignment with curriculum standards, and encouragement throughout the development process. We are especially thankful to **Mr. V. J. Moses, Head of Department, Basic Sciences & Humanities**, for his leadership, motivation, and valuable feedback during the preparation of this text. We are also indebted to our colleagues in the Department of Mathematics, whose peer review, suggestions, and insights have substantially enriched the clarity and correctness of the content. We likewise acknowledge the administrative and technical staff for their assistance in editing, formatting, and ensuring the timely publication of this material.

We anticipate that this text will serve as an indispensable companion to students and faculty alike, promoting a deeper appreciation of mathematical rigor and preparing students for onward challenges in their engineering journey.

Prepared by
Department of Mathematics / Course Team

Message from the Principal

Dr. R. Nagendra Babu

It gives me great pleasure to extend my greetings to all first-year B.Tech students as you embark on your engineering journey. Mathematics is a critical pillar in every branch of engineering, and in your formative years, a strong grounding in **Linear Algebra & Calculus** will equip you with the analytic tools and logical rigor needed for advanced study and innovation.

The material compiled in this text has been meticulously drafted to present concepts clearly, illustrate their relevance with examples, and offer problems that challenge and strengthen your understanding. I am confident that it will serve both as a guide and a companion through your semester, helping you grow in mathematical maturity and problem-solving ability.

I warmly acknowledge and thank the Department of Mathematics, under the leadership of **Mr. V. J. Moses, HOD, B.S. & H.**, and the academic support from **Dr. K. Kiran Kumar, Dean (Academics)**, for facilitating this work. Their commitment ensures that our students receive superior academic resources.

My hope is that as you engage with this text, you cultivate perseverance, intellectual curiosity, and discipline. I encourage you to use this as more than a textbook—as a stepping stone to critical thinking, innovation, and lifelong learning.

Wishing you success and fulfillment in your academic endeavors.

Dr. R. Nagendra Babu
Principal

Message from the Dean (Academics)

Dr. K. Kiran Kumar

I extend my warm greetings to all first-year B.Tech students as you begin this important stage of your academic journey. Mastery in mathematics furnishes an essential foundation for engineering studies, and competence in **Linear Algebra & Calculus** in particular will support your success in diverse disciplines—be it electronics, mechanics, computer science, or civil engineering.

This course material has been carefully compiled to present topics with clarity, integrate illustrative examples, and introduce exercises that will progressively challenge your understanding and problem-solving skills. I trust that it will serve as a reliable academic tool for both students and faculty.

I also take this opportunity to commend the Department of Mathematics, led by **Mr. V. J. Moses, HOD (B.S. & H.)**, for their dedicated efforts in content development, peer review, and pedagogical alignment. Their perseverance and scholarly integrity have elevated the quality of this text.

I encourage every student to engage with the material thoroughly, attempt each exercise, and not hesitate in seeking clarification when needed. Let curiosity guide you and discipline sustain you. May this textbook be a stepping stone towards deeper learning and academic excellence.

Dr. K. Kiran Kumar
Dean (Academics)

Why Do We Study Linear Algebra and Calculus in B.Tech First Year ?

1 Foundation for All Engineering Fields

Both **Linear Algebra** and **Calculus** form the *mathematical language* of engineering.
They help you describe and solve real-world problems that involve **change, motion, and systems**.

2 Linear Algebra – The Language of Systems**

Linear Algebra deals with **vectors, matrices, and linear equations** — which are everywhere in engineering.

Applications:

* □ Electrical Engineering: Solving circuit equations using matrices.
* □ Computer Science: Graphics, data representation, machine learning (AI).
* □ Civil/Mechanical Engineering: Structural analysis and modeling forces.
* □ Communication: Signal processing and image compression.

It helps you **model and analyze systems** efficiently.

3 Calculus – The Study of Change

Calculus teaches how quantities change and accumulate — fundamental to understanding how systems behave.

Applications:

* ⚙ Mechanical: Motion, velocity, acceleration, fluid flow.
* □ Electrical: Current, voltage changes, and optimization.
* □ Civil: Design curves, material strength, and load changes.
* □ Computer Science: Algorithms, optimization, and machine learning.

It gives tools for **designing, predicting, and optimizing** systems.

4 Builds Logical and Analytical Thinking

Both subjects train your brain to think in a **structured, logical, and problem-solving** way — essential for engineers.

5 Essential for Higher Courses

They are the **foundation** for later B.Tech subjects like:

* Numerical Methods
* Signals & Systems
* Control Systems
* Data Science / Machine Learning
* Structural Analysis
* Fluid Mechanics

Without Linear Algebra and Calculus, these advanced subjects would be difficult to grasp.

In short:

> **Linear Algebra helps you understand systems and structures.**
> **Calculus helps you understand change and motion.**
> Together, they make you *think like an engineer.*

L	T	P	C
3	0	0	3

LINEAR ALGEBRA & CALCULUS
(Common to All Branches of Engineering)

Course Objectives:

- To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real-world problems and their applications.

Course Outcomes: At the end of the course, the student will be able to
CO1: Develop and use of matrix algebra techniques that are needed by engineers for practical applications.
CO2: Relate eigen values and eigen vectors of a square matrix
CO3: Familiarize with functions of several variables which is useful in optimization.
CO4: Learn important tools of calculus in higher dimensions.
CO5: Familiarize with double and triple integrals of functions of several variables in two dimensions using Cartesian and polar coordinates and in three dimensions using cylindrical and spherical coordinates.

UNIT I Matrices
Rank of a matrix by echelon form, normal form. Cauchy–Binet formulae (without proof). Inverse of Non- singular matrices by Gauss-Jordan method, System of linear equations: Solving system of Homogeneous and Non-Homogeneous equations by Gauss elimination method. Jacobi and Gauss Seidel Iteration Methods.

UNIT II Eigenvalues, Eigenvectors and Orthogonal Transformation
Eigenvalues, Eigenvectors and their properties, Diagonalization of a matrix, Cayley-Hamilton Theorem (without proof), finding inverse and power of a matrix by Cayley-Hamilton Theorem, Quadratic forms and Nature of the Quadratic Forms, Reduction of Quadratic form to canonical forms by Orthogonal Transformation.

UNIT III Calculus
Mean Value Theorems: Rolle's Theorem, Lagrange's mean value theorem with their geometrical interpretation, Cauchy's mean value theorem, Taylor's and Maclaurin theorems with remainders (without proof), Problems and applications on the above theorems.

UNIT IV Partial differentiation and Applications (Multi variable calculus)
Functions of several variables: Continuity and Differentiability, Partial derivatives, total derivatives, chain rule, Taylor's and Maclaurin's series expansion offunctions of two variables. Jacobians, Functional dependence, maxima and minima of functionsof two variables, method of Lagrange multipliers.

UNIT V Multiple Integrals (Multi variable Calculus)
Double integrals, change of order of integration, triple integrals ,change of variables to polar, cylindrical and spherical coordinates. Finding areas (by double integrals) and volumes (by double integrals and triple integrals).

Textbooks:

1. Higher Engineering Mathematics, B. S. Grewal, Khanna Publishers, 2017, 44th Edition
2. Advanced Engineering Mathematics, Erwin Kreyszig, John Wiley & Sons, 2018, 10th Edition.

Reference Books:

1. Thomas Calculus, George B. Thomas, Maurice D. Weir and Joel Hass, Pearson Publishers, 2018, 14th Edition.
2. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar, Alpha ScienceInternational Ltd., 2021 5th Edition(9th reprint).
3. Advanced Modern Engineering Mathematics, Glyn James, Pearson publishers, 2018, 5th Edition.
4. Advanced Engineering Mathematics, Micheael Greenberg, , Pearson publishers, 9th edition
5. Higher Engineering Mathematics, H. K Das, Er. Rajnish Verma, S. Chand Publications,2014, Third Edition (Reprint 2021)

Prepared by
Ch. Someswara Rao
Assist. Professor of Mathematics

* Linear Algebra and Calculus *

Unit :- I

* Matrices *

Matrix Definition :-

A system of mn numbers (real and complex) arranged in the form of an order set of m rows each row consisting of an order set of n numbers between [ ] (or) ( ) (or) ‖ ‖ is called a matrix of order (or) type m×n.

Each of mn numbers constituting the m×n matrix is called an element of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m\times n}$$

$$A = [a_{ij}]_{m\times n}, \quad 1 \le i \le m, \ 1 \le j \le n$$

In relation of a matrix, we call the numbers as a scalars.

Unit matrix :- (or) Identity Matrix

If $[a_{ij}]_{n\times n}$ such that $a_{ij} = 1$ ~~and a_{ij}~~ for $i = j$ and $a_{ij} = 0$ for $i \neq j$ then A is called a "Unit matrix". It is denoted by I_n.

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Principle Diagonal :-

In a matrix $A = [a_{ij}]_{m\times n}$ the elements a_{ij} of A for which $i = j$ $[a_{11}, a_{22}, \cdots a_{nn}]$ are called diagonal element of A the line along which the diagonal is called the principle diagonal of A.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $A = \text{diag}(1, 6, 1)$

Scalar matrix :-

A diagonal matrix whose leading diagonal elements are equal and is called a Scalar matrix.

Ex :- $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Positive Integral powers of Square matrices :-

let ~~A, B~~ A be a square matrix then A^2 is defined as $(A \times A)$ now by

now By Associative law

$A^2 A = (A \cdot A) \cdot A = A(AA) = A \cdot A^2$

So that we can write $\boxed{A^2 \cdot A = AA^2 = A^3}$

similarly, we have $AA^{m-1} = A^{m-1}A = A^m$

where 'm' is a positive Integer.

Further we have $\boxed{\begin{array}{l} A^m \cdot A^n = A^{m+n} \\ (A^m)^n = A^{mn} \end{array}}$

Where m, n are positive integers.

where $\boxed{I^n = I}$

Trace of a square matrix :-

let $A = [a_{ij}]_{m \times n}$ then trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$ and it is denoted by $tr(A)$

thus $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

7. Transpose of a Matrix :-

The matrix obtained from any given matrix by interchangin its rows and columns is called transpose of A.

It is denoted by A' or A^T

If $A = [a_{ij}]_{m \times n}$ then the transpose of A is

$A' = [b_{ji}]_{n \times m}$ where $a_{ij} = b_{ji}$

$\boxed{\therefore (A')' = A}$

Determinants :-

Minors and Cofactors of a square matrix :-

let $A = [a_{ij}]_{n \times m}$ be a square matrix form A the elements of ith row and jth column deleted the determinent of $(n-1)$ rows matrix M_{ij} is called the minor of a_{ij} of A

* It is denoted by $|M_{ij}|$. the
* the signed minor $(-1)^{i+j} |m_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}

* Thus if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

$$|A| = a_{11} |M_{11}| - a_{12} |M_{12}| + a_{13} |M_{13}|$$
$$= a_{11} A_{11} - a_{12} A_{12} + a_{13} A_{13}$$

* Find minor and Cofactors of a matrix.

$$A = \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{2} & \overset{a_{13}}{3} \\ \underset{a_{21}}{4} & \underset{a_{22}}{5} & \underset{a_{23}}{0} \\ \underset{a_{31}}{1} & \underset{a_{32}}{2} & \underset{a_{33}}{1} \end{bmatrix}$$

Cofactor of $1 (a_{11})$ $= (-1)^{i+j} |M_{ij}|$

$= (-1)^{1+1} \begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix}$

$= (1) | 5(1) - (0)(2))$

$= 5 - 0$

$= 5.$

Cofactors of $2 (a_{12})$ $= (-1)^{i+j} |M_{ij}|$

$= (-1)^{1+2} \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$

$= -(4-0)$

$= -4$

Cofactors of $3 (a_{13})$ $= (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}$

$= (1)(8-5)$

$= 3$

Cofactor of 4 $(a_{21}) = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$

$= -(2-6)$

$= -(-4)$

$= 4$

Cofactor of 5 $(a_{22}) = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$

$= 1(1-3)$

$= -2$

cofactor of $(a_{23}) = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$= (-1)(2-2)$

$= 0$

Cofactor of 1 $(a_{31}) = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}$

$= 10-15$

$= -15$

Cofactor of 2 $(a_{32}) = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix}$

$= (-1)(0-12)$

$= 12$

cofactor of 1 $(a_{33}) = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

$= (5-8)$

$= -3$

$$A = \begin{bmatrix} 5 & -4 & 3 \\ 4 & -2 & 0 \\ -15 & 12 & -3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 5 & 4 & -15 \\ -4 & -2 & 12 \\ 3 & 0 & -3 \end{bmatrix}$$

Adjoint of a square matrix :-

Let A be a square matrix of order n The transpose of a matrix got from A by replacing the elements of ~~capit~~ A by corresponding co-factors is called adjoint of a matrix...

* It is denoted by 'adj A'

Inverse of a matrix :-

Let A be any square matrix B, it exists $AB = BA = I$ then B is called "inverse of A".

* It is denoted by A^{-1}

Symmetric Matrix :-

A square matrix $A = [a_{ij}]_{m \times n}$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j. thus A is a symmetric matrix (=)

$A = A'$ (or) $A' = A$

Skew symmetric matrix :-

A square matrix $A = [a_{ij}]_{m \times n}$ is said to be skew-symmetric matrix if $a_{ij} = a_{ji}$ for every i and j. Thus, A is a skew-symmetric matrix $A' = -A$ (or) $A = -A'$

Orthogonal Matrix :-

A square matrix A is said to be orthogonal if $AA' = A'A = I$

i.e; $A^T = A^{-1}$

1. $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$ is orthogonal ?

Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$

$A' = \begin{bmatrix} 2 & 4 & 3 \\ -3 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

$A \cdot A' = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ -3 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2\times2 + (-3)(-3) + (1)(1) & 2(4) + (-3)(3) + (1)(1) & 2(3) + (-3)(1) + (1)(4) \\ 4\times2 + (3)(-3) + (1)(1) & 4(4) + 3(3) + (1)(1) & 4(3) + 3(1) + 1(4) \\ 3(2) + 1(-3) + 4(1) & 3(4) + (1)(3) + 4(1) & 3(3) + 1(1) + 4(4) \end{bmatrix}$

$$\begin{bmatrix} 4+9+1 & 8-9+1 & 6-3+4 \\ 8-9+1 & 16+9+1 & 12+3+4 \\ 6-3+4 & 12+3+4 & 9+1+16 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 7 \\ 0 & 26 & 19 \\ 7 & 19 & 26 \end{bmatrix} \neq I_3$$

- which is not an orthogonal matrix.

Rank of a Matrix :-

If A is a non zero matrix we say that R is the rank of ~~matrix~~. 'A', if the following conditions are satisfied

(i) Every $(r+1)$th order minor of A is zero

(ii) there exist atleast one order of r th minor of A which is not zero

(iii) Rank of A is denoted by $\rho(A)$

Problems On Rank :-

1. Find the rank of the matrices.

a). $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$ (B) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Sol :- i) Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$$|A| = 1(48-40)-2(36-28)+3(30-28)$$
$$= 8-2(8)+3(2)$$
$$= 14-16$$
$$= -2 \neq 0$$

$\therefore$ Rank of $A = 3$

$\rho(A) = 3$.

B) Sol let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$$|A| = 3(4-4) - 1(-1)(-12+12) + 2(-6+6)$$
$$= 3(0) + 1(0) + 2(0)$$
$$= 0$$
$$|A| = 0$$
$$\rho(A) < 3$$
$$\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 6-6 = 0$$
$$\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4 = -8 \neq 0.$$

2. Find the rank of the matrix for the following

a) $\begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 5 & 2 & 4 \end{bmatrix}$

a). $-1\begin{vmatrix} 6 & 1 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ -5 & 3 \end{vmatrix} + 6\begin{vmatrix} 3 & 6 \\ -5 & 1 \end{vmatrix}$

$$= -1(18-1) - 0(9+5) + 6(3+30)$$
$$= -1(17) - 0(14) + 6(33)$$
$$= -17 + 198$$
$$= 181 \neq 0$$

$\therefore \rho(A) = 3$

Rank of $A = 3$

c. $2\begin{vmatrix} 4 & -2 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix}$

$$= 2(16+4) + 1(4+10) + 3(2-20)$$
$$= 2(20) + 1(14) + 3(-18)$$
$$= 40 + 14 - 54$$
$$= 54 - 54$$

$A = 0.$

$\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 8+1 = 9$ $\begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} = (2-12) = -10.$

$\begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix} = 2-20 = -18$ $\begin{vmatrix} 4 & -2 \\ 2 & 4 \end{vmatrix} = 16+4 = 20$

$\rho A = 2$

$\therefore$ Rank of $A = 2.$

b). $1\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix}$

$$= 1(24-25) - 2(18-20) + 3(15-16)$$
$$= 1(-1) - 2(-2) + 3(-1)$$
$$= -1 + 4 - 3$$
$$= -4 + 4$$

$A = 0$

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2$ $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10-12 = -2.$

$\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 15-16 = -1$ $\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24-25 = -1$

$\rho A = 2.$

$\therefore$ Rank of $A = 2.$

Echolon form of a Matrix :-

* Reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$ into echlon form and hence find its rank.

Sol:- Given that

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Rough

$2R_1 : 2 \quad 4 \quad 6 \quad 0$

$3R_1 : 3 \quad 6 \quad 9 \quad 0$

$4R_1 : 6 \quad 12 \quad 18 \quad 0$

$R_2 : R_2 - 2R_1$, $R_3 : R_3 - 3R_1$, $R_4 : R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 : R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 : R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in the echolon form

$\rho(A) = 3.$

H.W

2. Reduce the matrix

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$ into echolon form and hence find its rank.

$R_2 : R_2 + R_1 \quad R_3 : R_3 + 2R_1 \quad R_4 : R_4 - R_1$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

Rough

$2(-5) - 11(-1)$

$-10 + 11$

$R_3 : 2R_3 - 11R_2 \quad R_4 : R_4 + 2R_2$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 : 6R_4 + R_3$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The no of non zeroes $= 0$

$\therefore$ The rank of the matrix $= 4$

$\rho(A) = 4.$

3. Find rank by using echlon form from the following matrix.

$$\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Given matrix $A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$R_2 \longleftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 : R_2 + 2R_1 \quad R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 : 3R_3 + 2R_2 \quad R_4 : 3R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echlon form

$\rho(A) = 2$,

$\therefore$ The rank of the matrix is 2.

HW

4. $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

Reduction to Normal form :- (or) Canonical

Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ into normal form and find a rank.

let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

$\sim$

$R_2 : R_2 + 2R_1 \quad R_3 : R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$

$R_3 : 4R_3 + R_2$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$

Rough

$2R_1 : 2 \quad 4 \quad 2 \quad 0$

$R_1 : 1 \quad 2 \quad 1 \quad 0$

$4R_3 \quad 0 \quad -8 \quad 4 \quad -32$

$R_1 : 4R_1 - R_2$

$\sim \begin{bmatrix} 4 & 0 & -1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$

$C_3 : 4C_3 + C_1$

$\sim \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 8 & 20 & 0 \\ 0 & 0 & 36 & -32 \end{bmatrix}$

$C_3 : C_3 - C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 9 & -8 \end{bmatrix}$

$C_3/3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

$$\frac{C_2}{2}, \frac{C_3}{3}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

$$C_4 : C_4 + 8C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_2 : R_2 - 1/3 R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is in normal form

$$\rho(A) = 3$$

$$C_3 : C_3 - \frac{1}{3} C_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

$$C_4 : C_4 + 8C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is the normal form

$$\rho(A) = 3$$

2. Reduce normal form of a matrix $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Sol:- Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 4 \end{bmatrix}$

$R_2 : R_2 - 3R_1 \quad R_3 : R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$\frac{R_2}{7}, \frac{R_3}{9}$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$C_2 : C_2 + 3C_1, \quad C_3 : C_3 + C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 : R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$C_3 ; C_3 - C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

which is in the normal form $\rho(A) = 2$.

3. Reduce normal form of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 3 \end{bmatrix}$

$R_2 : R_2 - 2R_1$, $R_3 : R_3 - 3R_1$, $R_4 : R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 3 \end{bmatrix}$$

$C_3 : C_3 - 3C_1$, $C_2 : C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 3 \end{bmatrix}$$

$C_2 : C_2 + C_4$

$C_3 : C_3 + C_4$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & -1 & -8 & 3 \end{pmatrix}$$

$\frac{C_2}{-1}, \frac{C_3}{-1}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & 1 & 8 & 3 \end{bmatrix}$$

$C_2 : C_2 + C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 5 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 4 & 8 & 3 \end{bmatrix}$$

$C_2/4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 8 & 3 \end{bmatrix}$$

$C_3 : C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix}$$

$C_4 : C_4 - 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

$C_4 : C_4 - 2C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & -6 \end{bmatrix}$$

$R_4 : R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

$R_4/3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

~~$\therefore$ Rank = 4.~~

$R_4 : R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_4/-2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is in normal form

$\therefore$ Rank of $A = 4$.

$\rho(A) = 4$.

H.W

4. Reduce into normal form of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ and hence fin rank.

5. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

~~4sol let A =~~ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

~~systeting~~ cof

Cauchy - Binet Formula :-

The formula is identity for the determinent of the product of two rectangular matrices so that one product matrix is a square matrix.

Statement :-

let A and B be two matrix of size $m \times n$ or $n \times m$ respectively and $m \times n$ then $\det(AB) = \sum_{S=\binom{[n]}{m}} \det(A_{[m],S}) \det(B_{S,[m]})$

Here $[n]$ det notes the set

$\{1, 2, 3 \ldots n\}$ and $\binom{[n]}{n}$ for the set of m combinations of $[n]$

i.e; subsets of $[n]$ of site m.

there are $\binom{n}{m}$ of them

For $S = \binom{[n]}{m} = A_{[m],S}$ is for the $m \times n$ matrix where columns and the columns of A at indices forms S and $B_{S,[m]}$ for the matrix $m \times m$ whose rows are the rows of B at indices forms.

The inverse of a matrix by elementory Transformations (Gauss - Jordan method)

Working Rule :-

1. suppose A is a non-singular matrix of order n then $A = I_n A$.
2. We apply elementory raw operations only to the matrix A and the ~~pe~~ pre factor I_n of the R.H.S we will do ~~this~~ till get as equation of form $I_n = BA$ then obviously B is the inverse of A.

Problems

1. Find the matrix of $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ elementory row operations (Gauss - Jordon method).

Given $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

We write $A = I_3 A$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_3 : R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$$

$R_1 : R_1 + 2R_2$, $R_3 : R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$\frac{R_3}{8}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix}$$

$R_1 : R_1 - 2R_3$ $R_2 : R_2 - R_3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/8 & 6/8 & 4/8 \\ -1/8 & 3/8 & -2/8 \\ 1/8 & 5/8 & 2/8 \end{bmatrix} A$$

$\therefore I_3 = BA$

$B = A^{-1}$

$\therefore A^{-1} = 1/8 \begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

2. Find the inverse of the matrix A using elemantory operations (i.e; using Gauss Jordan method)

$$A = \begin{bmatrix} -1 & -3 & 3 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Given $A = \begin{bmatrix} -1 & -3 & 3 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & -3 & 3 & 1 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 : R_2 + R_1$, $R_3 : R_3 - 2R_1$ $R_4 : R_4 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & -7 & 4 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} A$$

$R_1 = 2R_1 + R_2$ $R_3 : 2R_3 : 7R_2$, $R_4 : R_4 + R_2$

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -6 & -13 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -1 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 : 3R_2 + R_3$, $R_4 : 6R_4 + R_3$, $R_3 : R_3 + 3R_2$

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -0 & -6 & \\ 0 & -6 & 0 & -10 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -4 & -8 & 2 & 0 \\ -7 & -11 & 2 & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 : R_1 + R_4$; $R_2 : R_2 - 10R_4$, $R_3 : R_3 - 13R_4$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 & 6 \\ 6 & 2 & -18 & -60 \\ 6 & -24 & -24 & -78 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

$R_1/2$, $R_2/-6$, $R_3/-6$ $R_4/-1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ -1 & 1/3 & 3 & 10 \\ -1 & -4 & 4 & 13 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

$I_4 = BA \Rightarrow$

$$A^{-1} = B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ -1 & 1/3 & 3 & 10 \\ -1 & 4 & 4 & 13 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

System of linear simultaneous Equations:-

1. write the following Equations in matrix form $AX=B$ and solve for X by finding A^{-1} whose $x+y-2z=3$; $2x-y+z=0$, $3x+y-z=0$.

Sol:- Given equations

$$x+y-2z=3$$
$$2x-y+z=0$$
$$3x+y-z=8.$$

We write $AX=B$.

where $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$

Consider $A = I_3 A$

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 : R_2 - 2R_1$ $R_3 : R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 5 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$R_3 : 3R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 5 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & -2 & 3 \end{bmatrix} A.$$

$R_1 : 3R_1 + R_2$; $R_2 : R_2 - R_3$, $R_3/5$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & -3 \\ -1 & -2/5 & \frac{3}{5} \end{bmatrix} A$$

$R_1 : R_1 + R_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{5} & \frac{3}{5} \\ 3 & 3 & -3 \\ -1 & -2/5 & 3/5 \end{bmatrix} A$$

$R_1/3 , R_2/-3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} A.$$

$I_3 = CA$

$A^{-1} = C$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0+0+\frac{8}{5} \\ -3-0+8 \\ -3-0+\frac{24}{5} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ 5 \\ -3+\frac{24}{5} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/5 \\ 5 \\ 9/5 \end{bmatrix}$$

$\therefore x = 8/5 , y = 5, z = 9/5$

2. __For Non Homogeneous System :-__

Consistent :-

The system $AX=B$ is ~~consti~~ consistent if and only if rank of A = rank of AB and it has a solution.

1. If the $\rho(A) = \rho(AB) = n$ then the system has unique solution.

   where n - unknown Variable.

2. If $\rho(A) = \rho(AB) < n$ then the system is consistent but their exist infinite number of solutions.

3. If the $\rho(A) \neq \rho(AB)$ then the system is inconsistent it has no solution.

1. show that the equations $x+y+z=4$, $2x+5y-5z=3$, $x+7y-7z=5$. are not consistent.

Given solutions

$$x+y+z=4$$
$$2x+5y-5z=3$$
$$x+7y-7z=5$$

can be expressed as $AX=B$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -5 \\ 1 & 7 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -5 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$$

$R_2 : R_2 - 2R_1 \quad R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -7 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

$\rho(A) = 3, \quad \rho(AB) = 3, \quad n = 3$

It ~~have~~ has unique solution and which is consistent

$$x+y+z = 4$$
$$3y-7z = -5$$
$$6z = 11$$

$6z = 11$

$z = 11/6$

$3y - 7z = -5$

$3y - 7(11/6) = -5$

$3y - \frac{77}{6} = -5$

$3y = \frac{-30+77}{6}$

$3y = \frac{47}{6}$

$$\boxed{y = \frac{47}{18}}$$

$x + y + z = 4$

$x + \frac{47}{18} + \frac{11}{6} = 4$

$x = \frac{4}{1} - \frac{47}{18} - \frac{11}{6}$

$= \frac{72 - 47 - 33}{18}$

$= -\cancel{8}^{4}/\cancel{18}_{9}$

$x = \frac{-4}{9}$

2. Examine the following equations are consistent or inconsistent

a) $x - 4y + 7z - 8$

$3x + 8y - 2z = 6$

$7x - 8y + 26z = 31$

b) $x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

$x - y + z = -1$

Sol. Given ~~so~~ equations are

$x - 4y + 7z = 8$

$3x + 8y - 2z = 6$

$7x - 8y + 26z = 31$

let $A = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 6 \\ 31 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

we consider the argumented matrix is

$[AB] = \begin{bmatrix} 1 & -4 & 7 & 8 \\ 3 & 8 & -2 & 6 \\ 7 & -8 & 26 & 31 \end{bmatrix}$

$R_2 : R_2 - 3R_1$, $R_3 : R_3 - 7R_1$

$\sim \begin{bmatrix} 1 & -4 & 7 & 8 \\ 0 & 20 & -23 & -18 \\ 0 & 20 & -23 & -25 \end{bmatrix}$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -4 & 7 & 8 \\ 0 & 20 & -23 & -18 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$\rho(A) = 2$, $\rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

Which has no solution and it is inconsistent

b). Given equations are.

Sol

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

we considered the argumented matrix is

$$[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 : R_2 - 3R_1$, $R_3 : R_3 - 2R_1$, $R_4 : R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & +5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_3 : 7R_3 - 6R_2$; $R_4 : 7R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & +5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 5 & 20 \end{bmatrix}$$

$R_4/5$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_4 : R_4 + R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3, \quad \rho(AB) = 3$

$n = 3$

$\rho(A) = \rho(AB) = r = n.$

The system of equation are consistent it has unique solution.

$$x + 2y - z = 3$$
$$-7y + 5z = -8$$
$$-z = -4$$
$$\Rightarrow \boxed{z = 4.}$$

$$-7y + 5z = -8$$
$$-7y + 5(4) = -8$$
$$-7y + 20 = -8$$
$$-7y = -8 - 20$$
$$-7y = -28$$
$$7y = 28$$
$$y = 28/7$$
$$\boxed{y = 4}$$

$$x + 2y - z = 4$$
$$x + 2(4) - 4 = 3$$
$$x + 8 - 4 = 3$$
$$x + 4 = 3$$
$$x = 3 - 4$$
$$\boxed{x = -1}$$

$\boxed{\therefore x = -1, \ y = 4, \ z = 4}$

3. For what values of λ the equations $x+y+z=1$; $x+2y+4z=\lambda$; $x+4y+10z=\lambda^2$ have a solution and solve them Completely in each case.

Given equations are

$$\left.\begin{aligned} x+y+z&=1 \\ x+2y+4z&=\lambda \\ x+4y+10z&=\lambda^2 \end{aligned}\right\} \longrightarrow (1)$$

system (1) can be expressed as a matrix form of $AX=B$ where

$$A=\begin{bmatrix}1&1&1\\1&2&4\\1&4&10\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad B=\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}$$

Argumented matrix

$$[AB]=\begin{bmatrix}1&1&1&1\\1&2&4&\lambda\\1&4&10&\lambda^2\end{bmatrix}$$

$R_2: R_2-R_1$; $R_3: R_3-R_1$

$$\sim\begin{bmatrix}1&1&1&1\\0&1&3&\lambda-1\\0&3&9&\lambda^2-1\end{bmatrix}$$

$R_3: R_3-3R_2$

$$\sim\begin{bmatrix}1&1&1&1\\0&1&3&\lambda-1\\0&0&0&\lambda^2-3\lambda+2\end{bmatrix}$$

$\therefore\ \rho(A)=2 \quad \rho(AB)=3$

But given the system has a solution it must be consictance so that

$$\lambda^2-3\lambda+2=0$$

$$\lambda^2-\lambda-2\lambda+2=0$$

$$\lambda(\lambda-1)-2(\lambda-1)=0$$

$$(\lambda-1)(\lambda-2)=0$$

$$\lambda-1=0 \quad \text{or} \quad \lambda-2=0$$

$$\lambda=1 \quad \text{or} \quad \lambda=2$$

$$\lambda=1,2$$

Case (i) $\lambda=1$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)=2, \ \rho(AB)=2 \ \ n=3$

Given equations are consistance and will have

$$\rho(A)=\rho(AB)=2<n$$

infinite no of solution

$$x+y+z=1$$

$$y+3z=0$$

$$n-r=3-2=1$$

let $z=k$

$z=k$ substitute in (2)

$$y+3z=0$$

$$y+3k=0$$

$$\boxed{y=-3k}$$

The values of y and z substitute that in (1)

$$x+y+z=1$$

$$x-3k+k=1 \Rightarrow$$

$x - 2k = 1$

$\boxed{x = 1 + 2k}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2k \\ -3k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Case (vii) $\lambda = 2$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 \quad \rho(AB) = 2 \quad n = 3$

$\therefore \rho(A) = \rho(AB) = r < n$

Given equation are consistant and will have infinite many solutions

$x + y + z = 1$ —— ②

$y + 3z = 1$ —— ③

let $z = k$

$z = k$, substitute in ③

$y + 3z = 1$

$y + 3k = 1$

$\boxed{y = 1 - 3k}$

The values of y and z substitute that in ②

$x + 1 - 3k + k = 1$

$x + \cancel{1} - 3k + k - \cancel{1} = 0$

$x - 2k = 0$

$\boxed{x = 2k}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ 1-3k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$\cancel{\lambda^2 - 3\lambda + 2 = 0}$

$\cancel{\lambda^2 - \lambda - 2\lambda + 2 = 0}$

4. If $a+b+c \neq 0$ show that the system of equation $-2x+y+z = a$, $x-2y+z = b$; $x+2y-2z = c$ has no solution. If $a+b+c = 0$ show that it has infinetly many solutions.

Given equations are

$$\left.\begin{aligned} -2x+y+z &= a \\ x-2y+z &= b \\ x+2y-2z &= c \end{aligned}\right\} \text{——— (1)}$$

system (1) can be expresed has matrix form of

$AX = B$ where

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Argumented matrix

$$[AB] \sim \begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 2 & -2 & c \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & -2 & c \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \end{bmatrix}$$

$R_2 : R_2 - R_1 \quad R_3 : R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -2 & c \\ 0 & -4 & 3 & b-c \\ 0 & 5 & -3 & a+2c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & c \\ 0 & -4 & 3 & b-c \\ 0 & 0 & \cancel{5+4} & 4a+5b+c \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -2 & c \\ 0 & -4 & 3 & b-c \\ 0 & 4 & 0 & 4a+4b+c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & c \\ 0 & -4 & 3 & b-c \\ 0 & 4 & 0 & 4(a+b+c) \end{bmatrix}$$

$R_3 / 4$

$$\sim \begin{bmatrix} 1 & 2 & -2 & c \\ 0 & 4 & 3 & b-c \\ 0 & 0 & 0 & a+b+c \end{bmatrix}$$

$\rho(A) = e \quad \rho(AB) = 3$

$\rho(A) \neq \rho(AB)$ if $a+b+c \neq 0$

It system has no solution

If $a+b+c = 0$

$\rho(A) = 2, \ \rho(AB) = 2, \ n = 3$

$\rho(A) = \rho(AB) = r < n$

$x + 2y - 2z = c$ —— ②

$-4y + 3z = b - c$ —— ③

$y = 0$ —— ④

$y = 0$ in eqn ②

$0 + \cancel{5b}\ 3z = b - c$

$3z = b - c$

$z = \frac{b-c}{3} \Rightarrow z = \frac{b}{3} - \frac{c}{3}$

The value of y and z in ②

$x + 2(0) - 2\left(\frac{b}{3} - \frac{c}{3}\right) = c$

$x - \frac{2b}{3} - \frac{2c}{3} - c$

$x = \frac{2b}{3} + c + \frac{2c}{3}$

$x = \frac{2b}{3} + \frac{5c}{3}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2b}{3} + \frac{5c}{3} \\ 0 \\ \frac{b}{3} - c/3 \end{bmatrix}$$

5. Solve the linear equations by matrix method H.W

a. $x+y+z=6$
$2x+3y-2z=2$
$5x+y+2z=13$

b. $x+y+4z=6$
$x+2y-2z=6$
$x+y+z=6$

c. $x+2y+2z=4$
$2x-y+3z=9$
$3x-y+z=2$.

6. Find the values of λ for which the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$. will have infi number of solutions solve them with the λ values.

Sol Given equations are

$$\left.\begin{aligned} 3x-y+4z&=3\\ x+2y-3z&=-2\\ 6x+5y+\lambda z&=-3 \end{aligned}\right\} \text{———} ①$$

System ① can be expressed in a matrix form

$AX=B$

$$A=\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B=\begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

The Argumented matrix

$$[AB]=\begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$$

$R_2: 3R_2-R_1$, $R_3: R_3-2R_1$

$$\sim\begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda-8 & -9 \end{bmatrix}$$

$R_3: R_3-R_2$

$$\sim\begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda+5 & 0 \end{bmatrix}$$

if $\lambda+5=0$

$\boxed{\lambda=-5}$

if $\lambda=-5$, $\rho(A)=2$ $\rho(AB)=2$, $n=3$

$\rho(A)=\rho(AB)=2=r<n$

Given equation have infinite no of solution

$\lambda=-5$ then

$$\sim\begin{bmatrix}3 & -1 & 4 & 3\\ 0 & 7 & -13 & -9\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$n-r=3-2=1$

let $3x-y+4z=3$ —— ①

$7y-13z=-9$ —— ②

$z=k$ —— ③

The value of $z=k$ in ③

$7y-13(k)=-9$

$7y-13k=-9$

$7y=-9+13k$

$y=-9+13k$

$y=\frac{13}{7}k-\frac{9}{7}$

The values of y and z substitute in ①

$3x-\left(\frac{13k}{7}-\frac{9}{7}\right)+4k=3$

$3x-\frac{13k}{7}+\frac{4k}{1}+\frac{9}{7}=3$

$3x+\frac{-13k+28k}{7}=3-\frac{9}{7}$

$3x+\frac{15k}{7}=\frac{21-9}{7}$

$3x+\frac{15k}{7}=\frac{12}{7}$

$3x=\frac{-15k}{7}+\frac{12}{7}$

$$x = \frac{-15k}{21} + \frac{12}{21}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{12}{21} - \frac{15k}{21} \\ \frac{-9}{7} + \frac{13k}{7} \\ k \end{bmatrix}$$

Consistency of System of Homogeneous Linear Equations

1. Solve $x+y-2z+3w=0$; $x-2y+z-w=0$; $4x+y-5z+8w=0$; $5x-7y+2z-w=0$.

Given equations are

$$\left.\begin{aligned} x+y-2z+3w &= 0 \\ x-2y+z-w &= 0 \\ 4x+y-5z+8w &= 0 \\ 5x-7y+2z-w &= 0 \end{aligned}\right\} \quad (1)$$

System (1) can be expressed in the form of a matrix

$AX = 0$

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 - R_1$; $R_3 : R_3 - 4R_1$; $R_4 : R_4 - 5R_1$

$$[AB] \sim \begin{bmatrix} 1 & 1 & -3 & 3 & 0 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & -12 & 12 & -16 & 0 \end{bmatrix}$$

$R_3 : R_3 - R_2$; $R_4 : R_4 - 4R_2$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(AB) = 2 \quad n = 4$

$\rho(A) = \rho(AB) = 2 = r < n$

It has infinite many solutions (or) non trivial solution.

let $\quad x + y - 2z + 3w = 0 \longrightarrow (2)$

$-3y + 3z - 4w = 0 \longrightarrow (3)$

$z = k_1 \longrightarrow (4)$

$w = k_2 \longrightarrow (5)$

The values of z and w in (3)

$$-3y + 3(k_1) - 4(k_2) = 0$$

$$-3y + 3k_1 - 4k_2 = 0$$

$$3y = 3k_1 - 4k_2$$

$$y = \frac{\cancel{3}k_1}{\cancel{3}} - \frac{4k_2}{3}$$

$$\boxed{y = k_1 - \frac{4k_2}{3}}$$

The values of y, z and w in (2)

$$x + \left(k_1 - \frac{4k_2}{3}\right) - 2(k_1) + 3(k_2) = 0$$

$$x + (k_1 - 2k_1) + \left(-\frac{4k_2}{3} + 3k_2\right) = 0$$

$$x - k_1 + \left(\frac{-4k_2 + 9k_2}{3}\right) = 0$$

$$x - k_1 + \frac{5k_2}{3} = 0$$

$$\boxed{x = k_1 - 5k_2/3}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} k_1 - 5k_2/3 \\ k_1 - 4k_2/3 \\ k_1 \\ k_2 \end{bmatrix}$$

2. solve $x + y - 3z + 2w = 0$, $2x - y + 2z - 3w = 0$, $3x - 2y + z - 4w = 0$, $-4x + y - 3z + w = 0$

Sol Given system of equations are

$$\left.\begin{aligned} x+y-3z+2w&=0\\ 2x-y+2z-3w&=0\\ 3x-2y+z-4w&=0\\ -4x+y-3z+w&=0 \end{aligned}\right\} \quad (1)$$

Given system of equations can be expressed as $AX=O$

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 - 2R_1 \quad R_3 : R_3 - 3R_1 \quad R_4 : R_4 + 4R_1$

$$\begin{bmatrix} 1 & 1 & -3 & 2 & 0 \\ 0 & -3 & 8 & -7 & 0 \\ 0 & -5 & 10 & -10 & 0 \\ 0 & 5 & -15 & 9 & 0 \end{bmatrix}$$

$R_3 : 3R_3 - 5R_2 \ ; \quad R_4 : 3R_4 + 5R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 & 0 \\ 0 & -3 & 8 & -7 & 0 \\ 0 & 0 & -10 & 5 & 0 \\ 0 & 0 & -5 & -8 & 0 \end{bmatrix}$$

$R_4 : 2R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 & 0 \\ 0 & -3 & 8 & -7 & 0 \\ 0 & 0 & -10 & 5 & 0 \\ 0 & 0 & 0 & -21 & 0 \end{bmatrix}$$

$\rho(A) = 4 \quad \rho(AB) = 4 \quad \rho(A{\cdot}B \quad n = 4$

$\rho(A) = \rho(AB) = 4 = r = n$

Thus the equations has trivial solution
$x = 0, y = 0, z = 0, w = 0.$

Solution of linear systems Direct Method
(Gauss Elimination method)

1. Solve the equations $2x+y+z=10$; $3x+2y+3z=18$; $x+4y+9z=16$ using gauss elimination method.

Sol Given equations are

$$\left.\begin{array}{l} 2x+y+z=10 \\ 3x+2y+3z=18 \\ x+4y+9z=16 \end{array}\right\} \text{—— (1)}$$

System (1) can be expressed in the form $AX=B$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

Argumented matrix

$$AB = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 : 2R_2 - 3R_1$, $R_3 : 2R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$R_3 : R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$2x + y + z = 10$ —— (2)
$y + 3z = 6$ —— (3)
$-4z = -20$ —— (4)

④ $\Rightarrow$ $+4z = +20$

$z = \frac{20}{4}$

$\boxed{z=5}$

③ $\Rightarrow$ $y+3z=6$

$y+3(5)=6$

$y+15=6$

$y=6-15$

$\boxed{y=-9}$

② $\Rightarrow$ $2x+y+z=10$

$2x-9+5=10$

$2x=10+9-5$

$2x=19-5$

$2x=14$

$x=\frac{14}{2}$

$\boxed{x=7}$

$\therefore x=7,\ y=-9,\ z=5.$

H.W

solve $3x+y-z=3$; $2x-8y+z=-5$, $x-2y+9z=8$ by Gauss elimination method.

Given equations are

$$\left.\begin{array}{l} 3x+y-z=3 \\ 2x-8y+z=-5 \\ x-2y+9z=8 \end{array}\right\} \text{①}$$

System ① can be expressed in the form $Ax=B$ where

$$A=\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B=\begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

Argumented matrix

$$AB=\begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$$

$\sim R_2 : 3R_2 - 2R_1 \quad R_3 : 3R_3 - R_1$

$$\sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & -7 & 28 & 21 \end{bmatrix}$$

$R_3 : \frac{R_3}{-7}.$

$$\sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & 1 & -4 & -3 \end{bmatrix}$$

$R_3 : 26R_3 + R_2$

$\frac{26 \times 3}{78}$

$-78-21$

$$\sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & 0 & -99 & -99 \end{bmatrix}$$

$3x+y-z=3$ —— (2)

$-26y+5z=-21$ —— (3)

$-\cancel{99}z=-\cancel{99}$ —— (4)

$\boxed{z=1}$ —

From eqn (3)

$-26y+5(1)=-21$

$-26y+5=-21$

$-26y=-21-5$

$-\cancel{26}y=-\cancel{26}$

$\boxed{y=1}$

From (4)

$3x+\cancel{y}-\cancel{z}=3$

$\cancel{3}x=\cancel{3}$

$\boxed{x=1}$

$\therefore x=1, y=1, z=1$

Jacobi's Iteration Method :-

1. Using Jacobi's Iteration method, solve system of equations
$10x+2y+z=9$, $x+10y-z=-22$, $-2x+3y+10z=22$

Given

$10x+2y+z=9$

$10x = 9-2y-z$

$x = \frac{1}{10}[9-2y-z]$ —— 1

$x+10y-z = -22$

$10y = -22-x+z$

$10y = z-22-x$

$y = \frac{1}{10}[z-x-22]$ —— 2

$-2x+3y+10z = 22$

$10z = 22+2x-3y$

$z = \frac{1}{10}[22+2x-3y]$ —— 3

Consider the initial solutions as $x=0, y=0, z=0$ substituting these values 1, 2, 3 respectively

$x^{(1)} = \frac{1}{10}[9-2(0)-0]$

$x^{(1)} = \frac{1}{10}[9]$

$x^{(1)} = \frac{9}{10}$

$\boxed{x^{(1)} = 0.9}$

$y^{(1)} = \frac{1}{10}[z-x-22]$

$y^{(1)} = \frac{1}{10}[0-0-22]$

$y^{(1)} = \frac{1}{10}[-22]$

$y^{(1)} = \frac{-22}{10}$

$\boxed{y^{(1)} = -2.2}$

$$z^{(1)} = \frac{1}{10}[22+2x-3y]$$

$$z^{(1)} = \frac{1}{10}[22+2(0)-3(0)]$$

$$z^{(1)} = \frac{1}{10}[22]$$

$$z^{(1)} = \frac{22}{10}$$

$$\boxed{z^{(1)} = 2.2}$$

Substitute,

$x = x^{(1)} = 0.9$, $y = y^{(1)} = -2.2$, $z = z^{(1)} = 2.2$ in ①, ②, ③ respectively

$$x^{(2)} = \frac{1}{10}[9-2y-z]$$

$$x^{(2)} = \frac{1}{10}[9-2(-2.2)-2.2]$$

$$x^{(2)} = \frac{1}{10}[9+4.4-2.2]$$

$$x^{(2)} = \frac{1}{10} \times 11.2$$

$$\boxed{x^{(2)} = 1.12}$$

$$y^{(2)} = \frac{1}{10}[z-x-22]$$

$$y^{(2)} = \frac{1}{10}[2.2-0.9-22]$$

$$y^{(2)} = \frac{1}{10}[-20.7]$$

$$\boxed{y^{(2)} = -2.07}$$

$$z^{(2)} = \frac{1}{10}[22+2x-3y]$$

$$z^{(2)} = \frac{1}{10}[22+2(0.9)-3(-2.2)]$$

$$z^{(2)} = \frac{1}{10}[22+1.8+6.6]$$

$$z^{(2)} = \frac{1}{10} \times 30.4$$

$$\boxed{z^{(2)} = 3.04}$$

Substitute

$x = x^{(2)} = 1.12$, $y = y^{(2)} = -2.07$ and $z = z^{(2)} = 3.04$ in

①, ②, ③ respectively

$$x^{(3)} = \frac{1}{10}[9-2y-z]$$

$$x^{(3)} = \frac{1}{10}[9-2(-2.07)-3.04]$$

$$x^{(3)} = \frac{1}{10} \times 10.1$$

$$\boxed{x^{(3)} = 1.01}$$

$$y^{(3)} = \frac{1}{10}[z-x-22]$$

$$y^{(3)} = \frac{1}{10}[3.04-1.12-22]$$

$$y^{(3)} = \frac{1}{10} \times -20.08$$

$$y^{(3)} = \frac{-20.08}{10}$$

$$\boxed{y^{(3)} = -2.008}$$

$$z^{(3)} = \frac{1}{10}[22+2x-3y]$$

$$z^{(3)} = \frac{1}{10}[22+2(1.12)-3(-2.07)]$$

$$z^{(3)} = \frac{1}{10} \times 30.45$$

$$\boxed{z^{(3)} = 3.045}$$

Substitute,

$x = x^{(3)} = 1.01$, $y = y^{(3)} = -2.008$ and $z = z^{(3)} = 3.045$ in ①, ②, ③ respectively.

$$x^{(4)} = \frac{1}{10}[9-2y-z]$$

$$x^{(4)} = \frac{1}{10}[9-2(-2.008)-3.045]$$

$$x^{(4)} = \frac{1}{10}[9.971]$$

$$\boxed{x^{(4)} = 0.9971}$$

$$y^{(4)} = \frac{1}{10}[z-x-22]$$

$$y^{(4)} = \frac{1}{10}[3.045-1.01-22]$$

$$y^{(4)} = \frac{1}{10} \times -19.965$$

$$\boxed{y^{4} = -1.9965}$$

$z^{(4)} = \frac{1}{10}[22+2x-3y]$

$z^{(4)} = \frac{1}{10}[22+2(1.01)-3(-2.008)]$

$z^{(4)} = \frac{1}{10} \times 30.044$

$\boxed{z^{(4)} = \frac{1}{10} \times 3.0044}$

Substitute,

$x = x^4 = 0.9971$, $y = y^4 = -1.9965$, $z = z^4 = 3.0044$ in ①,②,③

respectively

$x^{(5)} = \frac{1}{10}[9-2y-z]$

$x^{(5)} = \frac{1}{10}[9-2(-1.9965)-3.0044]$

$x^{(5)} = \frac{1}{10} \times 9.9886$

$\boxed{x^{(5)} = 0.99886}$

$y^{(5)} = \frac{1}{10}[z-x-22]$

$y^{(5)} = \frac{1}{10}[3.0044-0.9971-22]$

$y^{(5)} = \frac{1}{10} \times -19.9927$

$\boxed{y^{(5)} = -1.99927}$

$z^{(5)} = \frac{1}{10}[22+2x-3y]$

$z^{(5)} = \frac{1}{10}[22+2(0.9971)-3(-1.9965)]$

$z^{(5)} = \frac{1}{10} \times 29.9837$

$\boxed{z^{(5)} = 2.99837}$

Substitute,

$x = x^5 = 0.99886$, $y = y^5 = -1.99927$, $z = z^5 = 2.99837$

$x^{(6)} = \frac{1}{10}[9-2y-z]$

$x^{(6)} = \frac{1}{10}[9-2(-1.99927)-2.99837]$

$= \frac{1}{10} 10.00017$

$= 1.00001$

$$y = \frac{1}{10}[z - x - 22]$$

$$y(6) = \frac{1}{10}[2.99837 - 0.99886 - 22]$$

$$= \frac{1}{10} \times -20.00049$$

$$y(6) = -2.0000$$

$$z(6) = \frac{1}{10}[22 + 2x - 3y]$$

$$= \frac{1}{10} \times [22 + 2(0.9986) - 3(-1.99927)]$$

$$= \frac{1}{10} \times 29.9955$$

$$= 2.9995.$$

Substitute

$x = x^{(6)} = 1.0000$ $\quad y = y^{(6)} = -2.0000$ $\quad z = z^{(6)} = 2.9995$

$$x(7) = \frac{1}{10}[9 - 2y - z]$$

$$= \frac{1}{10}[9 - 2(-2.0000) - 2.9995]$$

$$= \frac{1}{10} \times 10.0005$$

$$= 1.0000$$

$$y(7) = \frac{1}{10}(z - x - 22]$$

$$= \frac{1}{10}(2.9995 - 1.0000 - 22)$$

$$= \frac{1}{10}(-20.0005)$$

$$= -2.0000$$

$$z(7) = \frac{1}{10}[22 + 2x - 3y]$$

$$= \frac{1}{10}[22 + 2(1.0000) - 3(-2.0000)]$$

$$= \frac{1}{10} \times 30$$

$$z^{(7)} = 3.$$

Hence

$\therefore x = 1, y = -2, z = 3$

H.W

The system of equations by Jacobi's method $10x+y-z=14.1719$
$x+10y+z=28.08$, $-x+y+z=35.61$

Gauss Seidal Iteration Method

1. Use gauss - seidal iteration method to solve the system
$10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=14$.

Sol: Given equation

$10x+y+z=12$

$10x=12-y-z$

$x=\frac{1}{10}[12-y-z]$ —— ①

$2x+10y+z=13$

$10y=13-2x-z$

$y=\frac{1}{10}[13-2x-z]$ —— ②

~~$2x+10y+z=13$~~

~~$10y=13-2x-z$~~

~~$y=\frac{13-2x-z}{10}$~~

~~$y=\frac{1}{10}[13-2x-z]$~~ —— ③

$2x+2y+10z=14$

$10z=14-2x-2y$

$z=\frac{1}{10}[14-2x-2y]$ —— ③

We start iteration $y=0$ and $z=0$ in eq ①

$x^{(1)}=\frac{1}{10}[12-0-0]$

$x^{(1)}=\frac{1}{10}\times12$

$=1.2$

putting $x=x^{(1)}=1.2$, $z=0$ in eqn ②

$y^{(1)} = \frac{1}{10}[13-2(1.2)-0]$

$= \frac{1}{10} \times 10.6$

$= 1.06$

putting $x = x^{(1)} = 1.2$ and $y = y^{(1)} = 1.06$ in eqn ③

$z^{(1)} = \frac{1}{10}[14-2(1.2)-2(1.06)]$

$z^{(1)} = \frac{1}{10} \times 9.48$

$z^{(1)} = 0.948$

Putting $y = y^{(1)} = 1.06$ and $z = z^{(1)} = 0.948$ in eqn ①

$x^{(2)} = \frac{1}{10}[12-1.06-0.948]$

$x^{(2)} = \frac{1}{10} \times 9.992$

$= 0.9992$

putting $x = x^{(2)} = 0.9992$ $z = z^{(1)} = 0.948$ in eqn ②

$y^{(2)} = \frac{1}{10}[13-2(0.9992)-0.948]$

$y^{(2)} = 1.0053$

putting $x = x^{(2)} = 0.9991$, $y = y^{(2)} = 1.0053$ in eqn ③

$z^{(2)} = \frac{1}{10}[14-2(0.9992)-2(1.0053)]$

$z^{(2)} = \frac{1}{10} \times 9.991$

$z^{(2)} = 0.9991$

putting $y = y^{(2)} = 1.0053$, $z = z^{(2)} = 0.9991$ in ①

$x = \frac{1}{10}[12-1.0053-0.9991]$

$= \frac{1}{10} \times 9.9956$

$x^{(3)} = 0.99956$

putting $x = x^{(3)} = 0.9995$ and $z^{(2)} = 0.9991$ in ②

$y^{(3)} = \frac{1}{10}[13-2(0.9995)-0.9991]$

$y^{(3)} = \frac{1}{10} \times 10.0019$

$= 1.0001$

putting $x = x^{(3)} = 0.9995$ & $y = 1.0001$ eqn (3)

$z^{(3)} = \frac{1}{10}[14 - 2(0.9995) - 2(1.0001)]$

$z^{(3)} = \frac{1}{10} \times 10.0008$

$= 1.0000$

putting $y = y^{(3)} = 1.0001$ and $z = z^{(3)} = 1.0000$

$\cancel{x^{(4)} = \frac{1}{10}[13 - 2 - (0.9999}$

$x^{(4)} = \frac{1}{10}[12 - 1.0001 - 1.0000]$

$x^{(4)} = \frac{1}{10} \times [9.9999]$

$= 0.9999$

putting $x = x^{(4)} = 0.9999$ and $z^{(3)} = 1.0000$ in (2)

$y^{(4)} = \frac{1}{10}[13 - 2(0.9999) - 1.0000]$

$y^{(4)} = \frac{1}{10} \times [10.0002 = 1.0000$

Putting $x = x^{(4)} = 0.9999$ & $y = y^{(4)} = 1.0000$

$z^{(4)} = \frac{1}{10}[14 - 2(0.9999) - 2(1.0000)]$

$= \frac{1}{10} \times 10.0002$

$= 1.0000$

Values	I interation	II interation	III interation	IV interatio
x	1.2	0.9992	0.9995	0.9999
y	1.06	1.0053	1.0001	1.0000
z	0.948	0.9991	1.0000	1.0000.

H.W :-

2. solve the system of equations using Gauss - seidal iteration method

$27x-6y-z=85$, $6x+15y+2z=72$, $x+y+54z=110$.

Given equations

$27x-6y-z=85$

$27x=85+6y+z$

$x=\frac{1}{27}[85+6y+z]$ — ①

~~$6x+15y+27=72$~~

~~$6x=72-15y-27$~~

~~$x=\frac{72-15y-27}{6}$~~

$6x+15y+2z=72$

$15y=72-6x-2z$

$y=\frac{1}{15}[72-6x-2z]$ — ②

$x+y+54z=110$

$54z=110-x-y$

$z=\frac{1}{54}[110-x-y]$ — ③

we start iteration from $y=0$ and $z=0$ in ①

$x^{(1)}=\frac{1}{27}[85+0+0]$

$=\frac{1}{27}(85)$

$=3.148$

put $x=x^{(1)}=3.14$ and $z=0$ in ②

$y=\frac{1}{15}[72-6(3.14)-2(0)]$

$=\frac{1}{15}[72-18.84]$

$=\frac{1}{15}\times 53.16$

$\boxed{y^{(1)}=3.544}$

put $x=x^{(1)}=3.148$ and $y^{(1)}=3.544$ in ③

$z^{(1)}=\frac{1}{54}[110-3.148-3.544]$

$z^{(1)}=\frac{1}{54}\times 103.308$

$\boxed{z^{(1)}=1.9131}$

put $y^{(1)}=3.544$ and $z^{(1)}=1.9131$

$x^{(2)}=\frac{1}{27}[85+6(3.544)+1.9131]$

$=\frac{1}{27}\times 108.1771$

$\boxed{x^{(2)}=4.0065}$

put $x^{(2)}=4.0065$ and $z^{(1)}=1.9131$ in

~~$y=\frac{1}{15}[110-4.0065$~~

$y^{(2)}=\frac{1}{15}[72-6(4.0065)-2(1.9131)]$

$\boxed{y^{(2)}=44.1348}$

put $x^{(2)}=4.0065$ and $y^{(2)}=44.1348$ in ③

$z=\frac{1}{54}[110-4.0065-44.1348]$

$\boxed{z^{(2)}=61.8587}$

put $y^{(2)}=44.1348$ and $z^{(2)}=61.8587$ in eqn ①

$x^{(3)}=\frac{1}{27}[85+6(44.1348)+61.8587]$

$x^{(3)}=411.6675$

put $x(3)$ and $z^{(2)}$ in eqn ②

$y^{(3)}=\frac{1}{15}[72-6(411.66)-2(61.858)]$

$=\frac{1}{15}$

Date :- 15/9/25

Unit - II

Eigen Values, Eigen Vectors and Orthogonal Transformation

Let $A = [a_{ij}]_{m \times n}$ matrix a non zero vector X is said to be characteristics vector of A If there exist a scalar λ such that $AX = \lambda X$, if $AX = \lambda x$ $(x \neq 0)$ we say that X is Eigen vector or characteristic vector of A corresponding to the Eigen values or characteristic vectors or values $\lambda(A)$.

Note :-

$|A - \lambda I| = 0$ is called the characteristic equation of A. This will be polynomial equation in λ of degree n.

Here A is $n \times n$ matrix (square matrix) and I is the $n \times n$ unit matrix

i.e; It should be ~~st~~ satisfied.

Problems

1. Find the Eigen values and Eigen vectors of the following matrix of the following matrix

(i) $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Given matrix

$$A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix}$$

$|A-\lambda I| = 0$

$|A-\lambda I| = (5-\lambda)[(6-\lambda)(7-\lambda)-(2)(2)] + 2[(-2)(7-\lambda)-(2)(0)] + 0$

$(5-\lambda)[42-6\lambda-7\lambda+\lambda^2-4] + 2[-14+2\lambda-0] = 0$

$(5-\lambda)[\lambda^2-13\lambda+38] - 28+4\lambda = 0$

$5\lambda^2-65\lambda+190-\lambda^3+13\lambda^2-38\lambda-28+4\lambda = 0$

$-\lambda^3+18\lambda^2-99\lambda+162 = 0$

$\lambda^3-18\lambda^2+99\lambda-162 = 0$

put $\lambda = 3$ in equation 1

$= (3)^3-18(3)^2+99(3)-162$

$= 27-18(9)+297-162$

$= 27-162+297-162$

$= \cancel{324}-\cancel{324}$

$\boxed{\therefore \lambda = 3}$

3	1	-18	99	-162
	0	+3	-45	+162
	1	-15	54	0

$(\lambda-3)(\lambda^2-15\lambda+54) = 0$

$(\lambda-3)(\lambda-6)(\lambda-9) = 0$

$\lambda-3=0 \quad \lambda-6=0 \quad \lambda-9=0$

$\lambda=3 \quad \lambda=6 \quad \lambda=9$

$\therefore \boxed{\lambda = 3,6,9}$

$\therefore$ $\lambda = 3,6,9$ are the characteristics of A (or) eigen values of A case (i) :

If $\lambda=3$ then $(A-\lambda I)X=0$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : \longrightarrow R_2+R_1$, $R_3/2$

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3-R_2$

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$\rho(A)=2$, $n=3$

$n-r=3-2=1$ (L.I.S)

$2x-2y=0$ —— ①

$y+2z=0$ —— ③

let $z=k$

put $z=k$ in eqn ③

$y+2k=0$

$\boxed{y=-2k}$

The values of 'y' and 'z' substitute in eqn ②

$2x-2(-2k)=0$

$2x+4k=0$

$2x=-4k$

$x=-2k$

$$x=\begin{bmatrix} -2k \\ -2k \\ k \end{bmatrix}=k\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

case (ii)

If $\lambda=6$ then $(A-\lambda I)X=0$

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2/2$

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2 \ , \ n = 3$

$n - r = 3 - 2 = 1$ (LIS

$-x - 2y = 0$ — ②

$2y + z = 0$ — ③

$z = k$

put $z = k$ eq^n ③

$2y + k = 0$

$2y = -k$

$\boxed{y = -k/2}$

put $y = -k/2$ in eq^n ②

$-x - \cancel{2}(-k/\cancel{2}) = 0$

$-x + k = 0$

$\boxed{x = k}$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} k \\ -k/2 \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$$

Case iii)

$\lambda = 9$

then $(A - \lambda I) x = 0$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : 2R_2 - R_1$

$$\begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2/4 \,,\ R_3 \to R_3/2$

$$\begin{bmatrix} -4 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 + R_2$

$$\begin{bmatrix} -4 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2 \quad n = 3$

$n - r = 3 - 2 = 1$ (L.I.S)

$-4x - 2y = 0$ —— ②

$-y + z = 0$ —— ③

$\boxed{z = k}$

the value of z substitute in ③

$-y + k = 0$

$-y = -k$

$\boxed{y = k}$

the value of y substitute in ②

$-4x - 2k = 0$

$-4x = 2k$

$x = \frac{2k}{-4}$

$\boxed{x = -k/2.}$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -k/2 \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

properties of eigen values :-

1. The sum of the eigen values of a square matrix is equal to its trace and product of the Eigen values is equal to its determinent.
2. If λ is an Eigen value of A corresponding to the Eigen vector 'x' then λ^n is eigen value of A^n corresponding to the Eigen vector 'x'
3. A square matrix "A" and its transpose A^T have the same eigen values.
4. If A and B are nxn matrix and if A is invertible then $A^{-1}B$ and BA^{-1} has same eigen values.
5. If $\lambda_1, \lambda_2 --- \lambda_2$ are the Eigen values of matrix A.
6. If $k\lambda_1 = k\lambda_2 ---- k\lambda_n$ are the eigen values of the matrix kA.
7. If λ is the eigen value of the matrix A then $\lambda + k$ is an eigen value of the matrix $A + KI$
8. If λ is an Eigen value of a non singular matrix of a corresponding to the Eigen vector "λ" then λ^{-1} is an Eigen value of A^{-1} and the corresponding Eigen value itself.

2. Find the characteristic roots and characteristic vectors of the following matrices.

1. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 4. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

① If 2, 3, 5 are the Eigen values of a matrix A, then find the eigen values of $2A^3+3A^2+5A+3I$

let $f(A) = 2A^3+3A^2+5A+3I$

put $A=2$ $f(2) = 2(2)^3+3(2)^2+5(2)+3$

$= 2(8)+3(4)+5(2)+3$

$= 16+12+10+3$

$= 41$

put $A=3$ $f(3) = 2(3)^3+3(3)^2+5(3)+3$

$= 2(27)+3(9)+15+3$

$= 54+27+18$

$= 99$

put $A=5$ $f(5) = 2(5)^3+3(5)^2+5(5)+3$

$= 2(125)+3(25)+25+3$

$= 250+75+25+3$

$= 353$

Hence the Eigen Values of this Equation is $2A^3+3A^2+5A+3I$ are 41, 99, 353.

② Find the Eigen Values of A^{-1}, if $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: the given matrix A is an upper triangular matrix. so, the eigen values of A are the diagonal elements 2, 4, 3

λ is eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{4}, \frac{1}{3}$.

$\text{Adj } A = \frac{|A|}{\lambda}$

$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{vmatrix}$

$= 2(12-0)-3(0-0)+4(0-0)$

$= 2(12)-3(0)+4(0)$

$= 24$

The eigen values of adj A are $\frac{\cancel{24}^{12}}{\cancel{2}}, \frac{\cancel{24}^{8}}{\cancel{3}}, \frac{\cancel{24}^{6}}{\cancel{4}}$ (or) 12, 8, 6.

3. For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2-6A+2I$

The matrix A is an upper triangular matrix

$\therefore \lambda = 1, 3, -2$ (i.e; diagonal elements)

Let $f(A) = 3A^3+5(A)^2-6A+2I$

put $A=1$ $= 3(1)^3+5(1)^2-6(1)+2I$

$= 3+5-6+2$

$= 4$

put $A=3$ $= 3(3)^3+5(3)^2-6(3)+2I$

$= 3(27)+5(9)-18+2$

$= 81+45-18+2$

$= 126+2-18$

$= 128-18$

$= 110.$

put $A=2$ $= 3(-2)^3+5(-2)^2-6(-2)+2$

$= 3(-8)+20+12+2$

$= -24+20+14$

~~$= -44+14$~~

~~$= -30.$~~

$= -4+14$

$= 10.$

The eigen values of $3A^3+5A^2-6A+2I$ are 4, 110, 10.

Diagonalisation of a matrix :-

Let A be a square matrix if there exist a non singular matrix 'P' and a diagonal matrix 'D' such that $P^{-1}AP=D$, the 'A' is said to be diagonalisable and 'D' is said to be a diagonal form (or) canonical form of a matrix A whose diagonal elements are the eigen values of A.

Working Rule :-

1. let A be the square matrix which is to be diagonalised.
2. Find the Eigen values of the matrix.
3. Find the Eigen vectors of the matrix.
4. check whether the eigen vectors are L.I (or) not. If the eigen vectors are L.I ($i \neq 0$) then the matrix is diagonalisable otherwise not.
5. Form model matrix $P = [X_1, X_2, X_3]$, where X_1, X_2, X_3 are eigen vectors of A.
6. Find the inverse of P.
7. Find the diagonal matrix $D = P^{-1}AP$.

Calculations of power of a matrix :-

we can obtain powers of a matrix by using diagonalisation let then non-singular matrix P can be formed such that

$$\boxed{D = P^{-1}AP}$$

$$D^2 = (P^{-1}AP).(P^{-1}AP)$$

$$= P^{-1}A.PP^{-1}.AP$$

$$= P^{-1}AIAP$$

$$= P^{-1}AAP$$

$$= P^{-1}A^2P$$

$$D^3 = P^{-1}A^3P$$

$$\vdots \qquad \vdots$$

$$D^n = P^{-1}A^nP \quad \text{——— (1)}$$

$$PD^nP^{-1} = A^n.$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right] - 1\left[(1-\lambda)-3\right] + 3\left[1-3(5-\lambda)\right] = 0$$

$$(1-\lambda)\left[5-5\lambda-\lambda+\lambda^2-1\right] - 1\left[-\lambda-2\right] + 3\left[1-15+3\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2-6\lambda+4\right] + \left[\lambda+2\right] + 3\left[3\lambda-14\right] = 0$$

$$(\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda) + (\lambda+2) - 42 + 9\lambda = 0$$

$$-\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$ Wrong

put $\lambda = -2$

$$= (-2)^3 - 7(-2)^2 + 36$$

problems :-

1. Diagnalize the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and also find A^4

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Eigen values of the matrix 'A' is in the form

$|A - \lambda I| = 0.$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(5-\lambda)(1-\lambda) - 1\right] - 1\left[(1-\lambda) - 3\right] + 3\left[(1) - 3(5-\lambda)\right] = 0$$

$$(1-\lambda)\left[5 - 5\lambda - \lambda + \lambda^2 - 1\right] - 1\left[-\lambda - 2\right] + 3\left[1 - 15 + 3\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2 - 6\lambda + 4\right] + \left[\lambda + 2\right] + 3\left[3\lambda - 14\right] = 0$$

$$(\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda) + (\lambda + 2) - 42 + 9\lambda = 0$$

$$-\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

put $\lambda = -2 \Rightarrow (-2)^3 - 7(-2)^2 + 36$

$= -8 - 28 + 36$

$= -36 + 36$

$\Rightarrow 0$

$$\boxed{\lambda = -2}$$

$$\begin{array}{r|rrrr} & 1 & -7 & 0 & 36 \\ -2 & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$$

$$(\lambda + 2)(\lambda^2 - 6\lambda - 3\lambda + 18) = 0$$

$$(\lambda + 2)\left[\lambda(\lambda - 6) - 3(\lambda - 6)\right] = 0$$

$$(\lambda + 2)\left[(\lambda - 6)(\lambda - 3)\right] = 0$$

$\lambda + 2 = 0$ $\quad$ $\lambda - 6 = 0$ $\quad$ $\lambda - 3 = 0$

$\boxed{\lambda = -2}$ $\quad$ $\boxed{\lambda = 6}$ $\quad$ $\boxed{\lambda = 3}$

$$A-\lambda I \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

$\lambda = -2, 3, 6$

Case (i)

If $\lambda = -2$ then

$(A-\lambda I)X_1 = 0$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$\begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rho(A) = 2, \; n = 3$

$n - r = 3 - 2 = 1$

$3x + y + 3z = 0$ — ①

$20y = 0$ — ② $\Rightarrow 20y = 0$ $\boxed{y = 0}$

Let $\boxed{z = k}$ — ③

The value of y and z substitute in ①

$3x + 0 + 3k = 0$

$\cancel{3}x = -\cancel{3}k$

$\boxed{x = -k}$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

$\lambda = -2, 3, 6$

<u>case (ii)</u> If $\lambda = 3$ then

$(A - \lambda I) X_2 = 0$

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + 3R_1$

$$\begin{pmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2/5$

$$\begin{pmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rho(A) = 2$, $n = 3$

$n - r = 3 - 2 = 1$

$-2x + y + 3z = 0$ —— ①

$y + z = 0$ —— ②

$\boxed{z = k}$ —— ③

sub $z = k$ in eqn ②

$y + k = 0$

$\boxed{y = -k}$

The values of y and z in eqⁿ ①

$$-2x-k+3k=0$$

$$-2x+2k=0$$

$$-\cancel{2}x=-\cancel{2}k$$

$$\boxed{x=k}$$

$$x_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}k\\-k\\k\end{bmatrix}$$

$$\therefore X_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

<u>Case (iii)</u> If $\lambda=6$ then

$$(A-\lambda I)X_3=0$$

$$\begin{bmatrix}-5&1&3\\1&-1&1\\3&1&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2\to 5R_2+R_1 \quad R_3\to 5R_3+3R_1$

$$\begin{bmatrix}-5&1&3\\0&-4&8\\0&8&-16\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2\to R_2/4 \quad R_3\to R_3/8$

$$\begin{bmatrix}-5&1&3\\0&-1&2\\0&1&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3:\ R_3+R_2$

$$\begin{bmatrix}-5&1&3\\0&-1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-5x+y+3z=0$ ——— (1)

$-y+2z=0$ ——— (2)

$\boxed{z=k}$ ——— (3)

the value of $z=k$ in eqn (2)

$-y+2k=0$

$\cancel{-}y=\cancel{-}2k$

$\boxed{y=2k}$

Substitute $y=2k$ and $z=k$ in (1)

$-5x+2k+3k=0$

$-5x+5k=0$

$-\cancel{5}x=\cancel{-5}k$

$\boxed{x=k}$

$$x_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ 2k \\ k \end{pmatrix}$$

$$x_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ The Eigen ~~val~~ vectors of A are

$$x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

let $P[x_1 \ x_2 \ x_3] = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$|P| = \det P = -1(-1-2)-1(0-2)+1(0+1)$

$= -(-3)-(-2)+1(1)$

$= 3+2+1$

$= 6.$

$$\operatorname{adj} P = \begin{bmatrix} +(-1-2) & -(0-2) & +(0+1) \\ -(1-1) & +(-1-1) & -(-1-1) \\ +(2+1) & -(-2-0) & +(1-0) \end{bmatrix}^T$$

$$\operatorname{adj} P = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}^T$$

$$\operatorname{adj} P = \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\operatorname{adj} P}{\det P} = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$D = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1+0+3 & 1-1+3 & 1+2+3 \\ -1+0+1 & 1-5+1 & 1+10+1 \\ -3+0+1 & 3-1+1 & 3+2+1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -6+0-6 & -9-0+9 & -18+0+18 \\ 4-0-4 & 6+6+6 & 12-24+12 \\ 2+0-2 & 3-6+3 & 6+24+6 \end{bmatrix}$$

$$D = \frac{1}{6}\begin{bmatrix} -12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

D is the diagnolisation of A.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & (3)^4 & 0 \\ 0 & 0 & (6)^4 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}$$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -48+0+0 & 0+0+0 & 48+0+0 \\ 0+162+0 & 0-162+0 & 0+162+0 \\ 0+0+1296 & 0+0+2592 & 0+0+1296 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -48 & 0 & 48 \\ 162 & -162 & 162 \\ 1296 & 2592 & 1296 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 48+162+1296 & 0-162+2592 & -48+162+1296 \\ 0-162+2592 & 0+162+5184 & 0-162+2592 \\ -48+162+1296 & 0-162+2592 & 48+162+1296 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1506 & 2430 & 1410 \\ 2430 & 5346 & 2430 \\ 1410 & 2430 & 1506 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

2. Determine the model matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ verify that is a diagonal matrix

The Caley Hamilton Theorem:-

Statement :-

Every square matrix satisfies its own characteristics equation

problems :-

1. If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify Caley Hamilton theorem. and Hence find A^{-1} and find A^4.

Given matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

The characteristic equation of matrix A

$(A-\lambda I)$

$$A-\lambda I = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(-2-\lambda)-0\right] - 1\left[5(-2-\lambda)+3\right] + 2\left[0-(3-\lambda)(-1)\right] = 0$$

$$(2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2\right] - \left[-10-5\lambda+3\right] + 2\left[3-\lambda\right] = 0$$

$(2-\lambda)[\lambda^2-\lambda-6]-[-5\lambda-7]+6-2\lambda=0$

$2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda+5\lambda+7+6-2\lambda=0$

$-\lambda^3+3\lambda^2+7\lambda+1=0$

$\boxed{\lambda^3-3\lambda^2-7\lambda-1=0}$

By caley Hamilton theorem

$A^3-3A^2-7A-I=0$

Here given $A=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$

$A^2=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$

$A^2=\begin{bmatrix}4+5-2&2+3+0&4+3-4\\10+15-3&5+9+0&10+9-6\\-2+0+2&-1+0+0&-2+0+4\end{bmatrix}$

$A^2=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$

$A^3=\begin{bmatrix}14+25-3&7+15+0&14+15-6\\44+70-13&22+42+0&44+42-26\\0-5-2&0-3+0&0-3-4\end{bmatrix}$

$=\begin{bmatrix}36&22&23\\101&64&60\\-7&-3&-7\end{bmatrix}$

$A^3-3A^2-7A-I=\begin{bmatrix}36&22&23\\101&64&60\\-7&-3&-7\end{bmatrix}-3\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}-7\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$\begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} - \begin{bmatrix} 21 & 15 & 9 \\ 66 & 42 & 39 \\ 0 & -3 & 6 \end{bmatrix} - \begin{bmatrix} 14 & 7 & 14 \\ 35 & 21 & 21 \\ -7 & 0 & -14 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence caley Hamilton Theorem is Verified.

Now, $A^3 - 3A^2 - 7A - I = 0$

$I = A^3 - 3A^2 - 7A$

Multiplying with A^{-1} on both sides

$A^{-1} = A^2 - 3A - 7I$

$$A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 3 & 6 \\ 15 & 9 & 9 \\ -3 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

consider $A^3 - 3A^2 - 7A - I = 0$

on both sides multiplying with A

$A[A^3 - 3A^2 - 7A - I] = A(0)$

$A^4 - 3A^3 - 7A^2 - AI = 0$

$A^4 = 3A^3 + 7A^2 + A$

$$A^4 = 3\begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} + 7\begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 108 & 66 & 69 \\ 303 & 192 & 180 \\ -21 & -9 & -21 \end{bmatrix} + \begin{bmatrix} 49 & 35 & 21 \\ 154 & 98 & 91 \\ 0 & -7 & 14 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 159 & 102 & 92 \\ 462 & 293 & 274 \\ -22 & -16 & -9 \end{bmatrix}$$

Home Work problems

Find the inverse of the following matrices by using C-H-T and also verify C-H-T. Hence find A^{-1}

1. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ 3. $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$ 4. $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ 6. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

2M sums

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ express $2A^5 - 3A^4 + A^2 - 4I$ as of linear polynomial in A.

Sol Given that

$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ characterstic of matrix A is

$$A - \lambda I = 0$$

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}.$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$[(3-\lambda)(2-\lambda)] - [(1)(-1)] = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0$$

By caley Hamilton theorem

$$A^2 - 5A + 7 = 0$$

$$A^2 = 5A - 7I$$

$$A^3 = 5A^2 - 7A$$

$$A^4 = 5A^3 - 7A^2$$

$$A^5 = 5A^4 - 7A^3.$$

NOW. $2A^5 - 3A^4 + A^2 - 4I = 0$

$$= 2[5A^4 - 7A^3] - 3[5A^3 - 7A^2] + [5A - 7I] - 4I = 0$$

$$= 10A^4 - 14A^3 - 15A^3 + 21A^2 + 5A - 7I - 4I = 0$$

$$= 10A^4 - 29A^3 + 21A^2 + 5A - 11I = 0$$

$$= 10[5A^3 - 7A^2] - 29[5A^2 - 7A] + 21[5A - 7I] + 5A - 11I$$

$$= 50A^3 - 70A^2 - 145A^2 + 203A + 105A - 147I - 11I + 5A = 0$$

$$= 50A^3 - 215A^2 + 313A - 158I = 0$$

$$= 50(5A^2 - 7A) - 215(5A - 7I) + 313A - 158I = 0$$

$$= 250A^2 - 350A - 1075A + 1505I + 313A - 158I = 0$$

$$= 250[5A - 7I] - 1112A + 1347I = 0$$

$= 1250A - 1750I - 1112A + 1347I = 0.$

2. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a polynomial.

Given that $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ characteristic of a matrix A is

$$A - \lambda I = 0$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$[(1-\lambda)(3-\lambda)] - [(2)(-1)] = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

By caley Hamilton theorem

$$A^2 - 4A + 5 = 0$$

$$A^2 = 4A - 5I$$

$$A^3 = 4A^2 - 5A$$

$$A^4 = 4A^3 - 5A^2$$

$$A^5 = 4A^4 - 5A^3$$

$$A^6 = 4A^5 - 5A^4$$

Now,

$$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = 0$$

$$4A^5 - 5A^4 I - 4(4A^4 - 5A^3) + 8(4A^3 - 5A^2) - 12(4A^2 - 5A) + 14(4A - 5) = 0.$$

~~$4A^5 - 5A^4 I - A$~~

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= [4A^5 - 5A^4] - 4[4A^4 - 5A^3] + 8[4A^3 - 5A^2] - 12[4A^2 - 5A] + 14[4A - 5I]$

$= 4A^5 - 21A^4 + 52A^3 - 88A^2 + 116A - 70I$

$= 4[4A^4 - 5A^3] - 21[4A^3 - 5A^2] + 52[4A^2 - 5A] - 88[$

$= 16A^4 - 104A^3 + 313A^2 - 496A + 370I$

$= 16[4A^3 - 5A^2] - 104[4A^2 - 5A] + 313[4A - 5I] - 496A + 370I$

$= 64A^3 - 496A^2 + 1276A - 1195I$

$= 64[4A^2 - 5A] - 496[4A - 5I] + 1276A - 1195I$

$= 256A^2 - 1028A + 1285I$

$= 256[4A - 5I] - 1028A + 1285I$

$= -4A + 5I$ //

Quadratic forms :-

A homogeneous expression of the second degree in any number of variables is called a Quadratic form.

Ex :- 1. $3x^2 + 5xy - 2y^2$ is a quadratic form in x & y.

2. $x^2 + y^2 - 3z^2 + 2xy - 8yz + 5zx$ is a quadratic form in three variables.

* An expression of the form $Q = \bar{X}'AX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ where a_{ij} are constants is called a Quadratic form in 'n' variable.

Matrix of a Quadratic form :

Every Quadrantic form Q can be expressed as $Q = X$. The symmetric matrix A is called the matrix of the Quadratic form Q and $|A|$ is called the discriminent of Quadratic.

form.

Problems :-

1. $Q: 7x^2+8xy+9yz+2xz+3y^2-5z^2$ write in symmetric matrix.

sol: Given,

$$Q = 7x^2+8xy+9yz+2xz+3y^2-5z^2.$$

$$Q = 7xx+4xy+4yx+\frac{9}{2}yz+\frac{9}{2}zy+xz+zx+3yy-5z.z$$

$$Q = \begin{array}{l} 7xx+4xy+xz \\ 4yx+3yy+\frac{9}{2}yz \\ zx+\frac{9}{2}zy-5zz \end{array}$$

	x	y	z
x	x^2	$\frac{xy}{2}$	$\frac{xz}{2}$
y	$\frac{yx}{2}$	y^2	$\frac{yz}{2}$
z	$\frac{zx}{2}$	$\frac{zy}{2}$	z^2

$$A = \begin{bmatrix} 7 & 4 & 1 \\ 4 & 3 & \frac{9}{2} \\ 1 & 9/2 & -5 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad X^T = [x\ y\ z]$$

$$Q = X^T A X$$

$$Q = [x\ y\ z] \begin{bmatrix} 7 & 4 & 1 \\ 4 & 3 & 9/2 \\ 1 & 9/2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

H.W

2. write the symmetrix matrix of the following quadratic form: is $x^2+2y^2-7z^2-4xy-6zx$

3. write the symmetric matrix of the following quadratic form $2x^2-3y^2+5z^2+6xy-yz+4zx$

4. write the symmetric matrix of the following Q.F:
$x^2-y^2+z^2+7xy+9yz+11zx$

2 sol Given

$$Q = x^2+2y^2-7z^2-4xy-6zx$$

$$Q = xx+2yy-7z.z-2xy-2yx-3zx-3xz$$

$$Q =$$

$xx - 2xy - 3xz$

$-2yx + 2yy + 0$

$-3zx + 0 + zz$

$$= \begin{bmatrix} 1 & -2 & -3 \\ -2 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad x^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

$Q = X^T A X$

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ -2 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2 sol :-

Given $Q = 2x^2 - 3y^2 + 5z^2 + 6xy - yz + 4zx$

$Q = 2xx - 3.yy + 5.zz + 3xy + \cancel{3yy}^{3yx} - yz + 2zx + 2xz.$

$2xx + 3xy + 2xz$

$3yx + 3yy - yz$

$2zx + 0 + 5zz$

$$= \begin{bmatrix} 2 & 3 & 2 \\ 3 & 3 & -1 \\ 2 & 0 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad X^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

$Q = X^T A X$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ 3 & 3 & -1 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

4 sol :-

Given $Q = x^2 - y^2 + z^2 + 7xy + 9yz + 11zx$

$Q = xx - yy + zz + \frac{7}{2}xy + \frac{7}{2}yx + \frac{9}{2}yz + \frac{9}{2}zy + \frac{11}{2}zx +$

$xx + \frac{7}{2}xy + \frac{11}{12}xz$

$\frac{7}{2}xy - yy + \frac{9}{2}zy$

$\frac{11}{2}zx + \frac{9}{2}zy + zz.$

$$\begin{bmatrix} 1 & \frac{7}{2} & \frac{11}{2} \\ \frac{7}{2} & -1 & \frac{9}{2} \\ \frac{11}{2} & \frac{9}{2} & 1 \end{bmatrix}$$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad x^T = [x \; y \; z]$

$Q = x^T A X$

$= [x \; y \; z] \quad 1 \quad \frac{7}{2} \quad \frac{11}{2}$

II write the quadratic form of corresponding to the matrix

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 5) $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

1 sol Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

Quadratic form $Q = X^T A X$

Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$x^T = [x \; y \; z]$

$Q = X^T A X$

$Q = (x \; y \; z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$Q = [\, x+2y+3z \quad 2x+0+3z \quad 3x+3y+z \; \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

5. $Q = 2xy + 3yz + 4zx$

$$Q = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 0 & 3 \\ 4 & 3 & 0 \end{bmatrix}$$

II Write the Quadratic form of corresponding to the matrix

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$ 4) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

5) $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Sol ①: Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \quad X^T = [x\ y\ z],\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^TAX$

$$= [x\ y\ z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x + 2y + 3z & 2x + 3z & 3x + 3y + z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x(x + 2y + 3z) + y(2x + 3z) + z(3x + 3y + z)$$

$$= x^2 + z^2 + 4xy + 6zx + 3zy$$

All the roots are positive. The Q.F is +ve definite.

(2) Discuss the nature of the Q.F $x^2+4xy+4xz-y^2+2yz+4z^2$

Sol: Given Q.F : $x^2+4xy+4xz-y^2+2yz+4z^2$

This can be written in matrix form

$[x\ y\ z]\ A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$

characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(-1-\lambda)(4-\lambda)-1] - 2(2(4-\lambda)-3)$$

$$+3(2(1)-(-1-\lambda)(3)) = 0$$

$\lambda.(\lambda^2-4\lambda-15) = 0$

$\Rightarrow \lambda = 0$ or $\lambda^2-4\lambda-15 = 0$

$\therefore \lambda = 0$

Then the given Q.F is indefinite.

(3) Find the nature of the Q.F $2x^2+2y^2+2z^2-2xy-2yz-2zx$

$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$\lambda = 0, 3, 3$

The one eigen value is zero and the other are positive. The given Q.F is semi definite.

(4) Reduce the Q.F $7x^2+6y^2+5z^2-4xy-4yz$ to the canonical form using Orthogonal Transformation

(or) Find Rank, Index and Signature of Q.F using diagonalization method $7x^2+6y^2+5z^2-4xy-4yz$.

Sol: Given Q.F is $7x^2+6y^2+5z^2-4xy-4yz$

$\Rightarrow 7xx+6yy+5zz-2xy-2xy-2yz-2yz$

	x	y	z
x	x^2	$\frac{xy}{2}$	$\frac{xz}{2}$
y	$\frac{yx}{2}$	y^2	$\frac{yz}{2}$
z	$\frac{zx}{2}$	$\frac{zy}{2}$	z^2

$\Rightarrow 7x^2 - 2xy + 0\cdot xz$

$-2yx + 6y^2 - 2yz$

$+0\cdot zx - 2zy + 5z^2$

Its matrix is given by

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

We write $A = I_3 \cdot A I_3$

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 7R_2 + 2R_1$

$$\begin{bmatrix} 7 & -2 & 0 \\ 0 & 38 & -14 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 7C_2 + 2C_1$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 266 & -14 \\ 0 & -14 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 19R_3 + R_2$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 266 & -14 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 2 & 7 & 19 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 19C_3 + C_2$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 266 & 0 \\ 0 & 0 & 1539 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 2 & 7 & 19 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 2 \\ 0 & 7 & 7 \\ 0 & 0 & 19 \end{bmatrix}$$

$\frac{R_1}{\sqrt{7}}, \frac{C_1}{\sqrt{7}}, \frac{R_2}{\sqrt{266}}, \frac{C_2}{\sqrt{266}}, \frac{R_3}{\sqrt{1539}}, \frac{C_3}{\sqrt{1539}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 & 0 \\ \frac{2}{\sqrt{266}} & \frac{7}{\sqrt{266}} & 0 \\ \frac{2}{\sqrt{1539}} & \frac{7}{\sqrt{1539}} & \frac{19}{\sqrt{1539}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{266}} & \frac{2}{\sqrt{1539}} \\ 0 & \frac{7}{\sqrt{266}} & \frac{7}{\sqrt{1539}} \\ 0 & 0 & \frac{19}{\sqrt{1539}} \end{bmatrix}$$

The canonical form the given Q.F is

$$Y^T D Y = [y_1 \ y_2 \ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + y_2^2 + y_3^2$$

The normal form of the Q.F is $y_1^2 + y_2^2 + y_3^2$

Rank $r = 3$, $n = 3$, $S = 3$.

(2) Reduce the Q.F $3x^2 + 3y^2 + 3z^2 + 4xy + 4yz + 8xz$ into canonical form by diagonalization. Find its nature, rank, Index and Signature

Sol: The matrix of the Q.F is

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 3 & 4 \\ 4 & 4 & 3 \end{bmatrix}$$

The characteristic equation of A is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & 3-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} 1-\lambda & -(1-\lambda) & 0 \\ 2 & 3-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)([(3-\lambda)(3-\lambda) - 16] + 1[2(3-\lambda) - 16] + 0) = 0$

$\Rightarrow (1-\lambda)[9 + \lambda^2 - 6\lambda - 16 - 2\lambda - 10] = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 8\lambda - 17) = 0$

$\lambda = 1$, $\quad \lambda = \frac{8 \pm \sqrt{64+68}}{2} = \frac{8 \pm \sqrt{132}}{2}$

$= 4 \pm \sqrt{33}$

The roots are $1, 4+\sqrt{33}, 4-\sqrt{33}$

The roots are real and different

The diagonal matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4+\sqrt{33} & 0 \\ 0 & 0 & 4-\sqrt{33} \end{bmatrix}$$

canonical form is $Y^T D Y$ i.e

$y_1^2 + (4+\sqrt{33})(y_2^2) + (4-\sqrt{33}) y_3^2$.

rank $= r = 3$

Index $= s =$ No. of positive terms in the normal form $= 3$

Signature $=$ No. of positive terms $-$ No. of negative term $= 3 - 0 = 3$

Since all the roots are positive, the Q.F is positive definite.

(3) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ diagonalization by orthogonal transformation. Find rank, Index and Signature

Sol. Given $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

The characteristic equation of A is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

1/2/18

Diagonalisation by orthogonal transformation

~~sappdse~~ suppose, a is a real symetry matrix then, a characteristis matrix of a will not be linearly indipendent. and also waise orthogonal. if we normalise each characteristics vector or eign vectors (X) we devide each component of X by the square root of the sum of the squares of all elements, write all normalized eign vectors to form normalized ~~motn~~ model matrix B then it can be easily show that B is an orthogonal matrix and:

B^{-1} equal to B transpose.

therefore the symelerity transform

$$B^{-1}AB = D$$

Where D is the diagonal matrix.

this tranformation ~~B~~ transpose AB is equal to D, is known as orthogonal transformation.

Q. Calculation of powers of a matrix.

let, A be the given matrix of ~~AB~~ order 3.

We know that

$$\begin{bmatrix} 0 & 0 & \lambda_3 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by

sol^n!. The CH matrix of A is

$A - \lambda I$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix}$$

The CH. eqn of A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)^2 - 4] - 1[(1-\lambda) - 2] - 1[-2-(-1+\lambda)]$$

$$(2-\lambda)[1 - 2\lambda + \lambda^2 - 4] - 1[-\lambda - 1] - 1[-2 + 1 - \lambda] = 0$$

$$(2-\lambda)(\lambda^2 - 2\lambda - 3] + \lambda + 1 + 1 + \lambda = 0$$

$$2\lambda^2 - 4\lambda - 6 - \lambda^3 + 2\lambda^2 + 3\lambda + 2\lambda + 2 = 0$$

$$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda - 4\lambda + 4 = 0$$

$$\lambda^2(\lambda - 1) - 3\lambda(\lambda - 1) - 4(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 3\lambda - 4) = 0$$

$$\lambda^2 - 4\lambda + \lambda - 4 = 0$$

$$\lambda(\lambda - 4) + 1(\lambda - 4) = 0$$

$$(\lambda + 1)(\lambda - 4) = 0$$

$$\lambda = -1, 1, 4.$$

$$\boxed{\lambda = -1, 1, 4}$$

Case (ii)

if $\lambda = -1$, then

$A - \lambda I = 0$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x + y - z = 0$

$5y - 5z = 0$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

Let,

$z = K$

$5y - 5K = 0$

$5y = 5K$

$y = K$

$3x + K - K = 0$

$x = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ K \\ K \end{bmatrix} = K \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case ②

if $\lambda = 1$ then

$$A - \lambda I = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y - z = 0$$
$$-y - z = 0$$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

Let, $z = k_1$

$$-y - k_1 = 0$$
$$-y = k_1$$
$$y = -k_1$$

$$x - k_1 - k_1 = 0$$
$$x - 2k_1 = 0$$
$$x = 2k_1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case ③ if $\lambda = 4$ then

$$A - \lambda I = 0$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y - z = 0$$

$$-5y - 5z = 0$$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

let

$$z = k_1$$

$$-5y - 5k_1 = 0$$

$$-5y = 5k_1$$

$$y = -k_1$$

$$-2x - k_1 - k_1 = 0$$

$$-2x - 2k_1 = 0$$

$$-2x = 2k_1$$

$$x = -k_1$$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

eign vector

$B = [x_1 \ x_2 \ x_3]$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$\frac{0}{\sqrt{0^2+1^2+1^2}}$ $\frac{2}{\sqrt{4+1+1}}$ $\frac{1}{\sqrt{1^2+1^2+1^2}}$

$\frac{1}{}$

$B^T = B^{-1}$

We observed that eigen vectors are pair-wise orthogonal.

$$B = \begin{bmatrix} 0 & \frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$\frac{0}{\sqrt2}$ $\frac{1}{\sqrt2}$

$\frac{2}{\sqrt6}$ $\frac{1}{\sqrt6}$

$$B^{-1} = B^T = \begin{bmatrix} 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$$= \begin{bmatrix} 0+\frac{1}{\sqrt2}-\frac{1}{\sqrt2} & 0+\frac{1}{\sqrt2}-\frac{2}{\sqrt2} & 0-\frac{2}{\sqrt2}+\frac{1}{\sqrt2} \\ \frac{4}{\sqrt6}-\frac{1}{\sqrt6}-\frac{1}{\sqrt6} & \frac{2}{\sqrt6}-\frac{1}{\sqrt6}-\frac{2}{\sqrt6} & \frac{-2}{\sqrt6}+\frac{2}{\sqrt6}+\frac{1}{\sqrt6} \\ \frac{2}{\sqrt3}+\frac{1}{\sqrt3}+\frac{1}{\sqrt3} & \frac{1}{\sqrt3}+\frac{1}{\sqrt3}+\frac{2}{\sqrt3} & \frac{-1}{\sqrt3}-\frac{2}{\sqrt3}-\frac{1}{\sqrt3} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$$\begin{bmatrix} 0-\frac{1}{2}-\frac{1}{2} & 0+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} & 0-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}} \\ 0-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} & \frac{4}{6}+\frac{1}{6}+\frac{1}{6} & \frac{2}{\sqrt{18}}-\frac{1}{\sqrt{18}}-\frac{1}{\sqrt{18}} \\ 0+\frac{4}{\sqrt{6}}-\frac{4}{\sqrt{6}} & \frac{8}{\sqrt{18}}-\frac{4}{\sqrt{18}}-\frac{4}{\sqrt{18}} & \frac{4}{3}+\frac{4}{3}+\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D(-1, 1, 4)$$

$$= \begin{bmatrix} 0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} & 0 + \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 - \frac{1}{2} - \frac{1}{2} & 0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} & 0 - \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \\ 0 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{4}{6} + \frac{1}{6} + \frac{1}{6} & \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}} \\ 0 + \frac{4}{\sqrt{6}} - \frac{4}{\sqrt{6}} & \frac{8}{\sqrt{18}} - \frac{4}{\sqrt{18}} - \frac{4}{\sqrt{18}} & \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D(-1, 1, 4)$$

$\therefore$ The reduced canonical form is $(-y_1^2 + y_2^2 + 4y_3^2)$

The orthogonal transformation is $X = BY$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{6} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{6} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{2}{\sqrt{6}} y_2 + \frac{1}{\sqrt{3}} y_3$$

$$x_2 = \frac{1}{\sqrt{2}} y_1 - \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{3}} y_3$$

$$x_3 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{6}} y_2 - \frac{1}{\sqrt{3}} y_3$$

$r = 3$, Index $(p) = 2$, Signature $= 2 - 1 = 1$