

Sequence and Series and Mean

Value Theorem.

Mean Value Theorem:

- (1) Rolle's theorem
- (2) Lagrange's theorem
- (3) Cauchy's theorem
- (4) Taylor's theorem
- (5) Maclaurin's theorem

(1) Rolle's Theorem:

* Verify the Rolle's theorem for the following functions.

- (1) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$
- (2) $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a, b]$
- (3) $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$
- (4) $f(x) = |x|$ in $[-1, 1]$
- (5) $f(x) = \frac{1}{x^2}$ in $[-1, 1]$
- (6) $f(x) = \sin x$ in $[-\pi, \pi]$
- (7) $f(x) = \tan x$ in $[0, \pi]$
- (8) $f(x) = \sec x$ in $[0, 2\pi]$
- (9) $f(x) = e^x \cdot \sin x$ in $[0, \pi]$
- (10) $f(x) = (x-a)^m \cdot (x-b)^n$ in $[a, b]$

Rolle's Theorem:

Let $f(x)$ be a function of x defined in (a, b)

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is derivable in (a, b)

(iii) $f(a) = f(b)$

then $\exists c \in (a, b) \cdot \exists f'(c) = 0$.

$$\textcircled{1} \quad f(x) = \frac{\sin x}{e^x} \quad [0, \pi]$$

(i) $f(x) = \frac{\sin x}{e^x}$ is continuous for all x !

$f(x)$ is continuous in $[0, \pi]$

$$\begin{aligned} \Rightarrow f'(x) &= \frac{e^x \cdot \cos x - \sin x \cdot e^x}{(e^x)^2} \\ &= \frac{e^x (\cos x - \sin x)}{(e^x)^2} \end{aligned}$$

$$f'(x) = \frac{\cos x - \sin x}{e^x} \text{ is exist } \forall x.$$

$\Rightarrow f'(x)$ is exist in the interval $[0, \pi]$

$\therefore f(x)$ is derivable in $(0, \pi)$.

$\Rightarrow$ We have to show that $f(0) = f(\pi)$

$$f(0) = \frac{\sin 0}{e^0} = \frac{0}{1} = 0.$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0.$$

Then $\exists$ exist $c \in (a, b) \ni f'(c) = 0$.

$$f(x) = \frac{\sin x}{e^x}$$

$$f'(x) = \frac{\cos x - \sin x}{e^x}$$

$$f'(c) = \frac{\cos c - \sin c}{e^c} = 0$$

$$\cos c - \sin c = 0$$

$$\sin c = \cos c$$

$$\tan c = 1$$

$$c = \tan^{-1}(1)$$

$$\boxed{c = \pi/4} \in [0, \pi]$$

$$\textcircled{2} \quad f(x) = \log \left(\frac{x^2 + ab}{x(a+b)} \right) \text{ in } [a, b] \quad a > 0, b > 0.$$

$f(x) = \log(x^2 + ab) - \log x(a+b)$ is continuous $\forall x$ except at $x=0 \notin [a, b]$.

(i) $f(x)$ is continuous in $[a, b]$

$$\begin{aligned} \text{(ii)} \quad f(x) &= \log(x^2 + ab) - \log x - \log(a+b) \\ &= \frac{1}{x^2 + ab} (2x) - \frac{1}{x} - 0. \end{aligned}$$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} \text{ is exist } (\forall x) \text{ in } (a,b)$$

$f(x)$ is derivable in (a,b)

$$\begin{aligned} f(a) &= \log(a^2+ab) - \log a(a+b) \\ &= \log(a^2+ab) - \log(a^2+ab) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(b) &= \log(b^2+ab) - \log b(a+b) \\ &= \log(b^2+ab) - \log(ab+b^2) \\ &= 0 \end{aligned}$$

$$f(a) = f(b)$$

$$\exists a, c \in (a,b) \Rightarrow f'(c) = 0$$

$$\text{We have } f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$$

$$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0$$

$$2c^2 - c^2 - ab = 0$$

$$c^2 = ab$$

$$c = \pm\sqrt{ab}$$

$$c = \sqrt{ab} \text{ (or)} -\sqrt{ab}$$

$$\boxed{c = \sqrt{ab}} \in (a,b)$$

$$(3) f(x) = x \cdot (x+3) e^{-x/2} \text{ in } [-3,0]$$

soln $f(x)$ is continuous $\forall x$.

(i) $f(x)$ is continuous in $[-3,0]$

$$(ii) f'(x) = (x^2+3x) e^{-x/2}$$

$$\begin{aligned} f'(x) &= (x^2+3x) e^{-x/2} \cdot \frac{-1}{2} + e^{-x/2} (2x+3) \\ &= -\frac{(x^2+3x)}{2} e^{-x/2} + (2x+3) e^{-x/2} \end{aligned}$$

$$= e^{-x/2} \left[(2x+3) - \frac{(x^2+3x)}{2} \right]$$

$$= e^{-x/2} \left[\frac{4x+6-x^2-3x}{2} \right]$$

$$= e^{-x/2} \left[\frac{-x^2+x+6}{2} \right]$$

$$= \frac{e^{-x/2}}{2} (-x^2+x+6)$$

$f'(x)$ is exist in $[-3,0]$.
 $\Rightarrow f(x)$ is derivable in $[-3,0]$.

We have to show that $f(-3) = f(0)$

$$\begin{aligned} f(-3) &= -3(-3+3)e^{-3/2} \\ &= -3(0)e^{-3/2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(0) &= 0(0+3)e^{-0/2} \\ &= 0 \end{aligned}$$

$$\boxed{\therefore f(-3) = f(0)}$$

Then $\exists a \in (a,b) \ni f'(c) = 0$.

$$f'(x) = \frac{e^{-x/2}}{2} (-x^2 + x + 6)$$

$$f'(c) = \frac{e^{-c/2}}{2} (-c^2 + c + 6) = 0$$

$$(-c^2 + c + 6)e^{-c/2} = 0$$

$$e^{-c/2} = 0 \quad \text{and} \quad -c^2 + c + 6 = 0 \quad \checkmark$$

$$c^2 - c - 6 = 0$$

$$c^2 - 3c + 2c - 6 = 0$$

$$c(c-3) + 2(c-3) = 0$$

$$(c-3)(c+2) = 0$$

$$c = 3, \quad \boxed{c = -2} \in (-3, 0)$$

(4) $f(x) = |x|$ in $[-1, 1]$

Sol:

We know that $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$

(i) $f(x) = |x|$ is continuous $\forall x$

$\Rightarrow |x|$ is continuous in $[-1, 1]$

(ii) The derivative of $|x|$ does not exist.

Because,

$$\begin{aligned} \text{L.H.D} \quad \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\ = \lim_{x \rightarrow 0^-} \frac{f(x-0)}{x} \end{aligned}$$

$$= \lim_{x \rightarrow 0^-} \frac{-x}{x}$$

$$= \lim_{x \rightarrow 0^-} (-1) = \underline{\underline{-1}}$$

R.H.D $\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \rightarrow 0^+} \frac{x - 0}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x}$$

$$= \lim_{x \rightarrow 0^+} (1) = 1$$

$$\boxed{\therefore \text{L.H.D} \neq \text{R.H.D}}$$

Hence Rolle's theorem is not verified.

(10). $f(x) = (x-a)^m \cdot (x-b)^n$ in $[a, b]$.

Sol: $f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[a, b]$

$\Rightarrow f'(x) = (x-a)^m \cdot (x-b)^n$

$$f'(x) = (x-a)^m \cdot n(x-b)^{n-1}(1-0) + (x-b)^n m(x-a)^{m-1}(1-0)$$

$$= n \cdot (x-b)^{n-1} \cdot (x-a)^m + m \cdot (x-a)^{m-1} \cdot (x-b)^n$$

$$= n \cdot (x-b)^{n-1} \cdot (x-b)^{-1} (x-a)^m + m \cdot (x-a)^{m-1} \cdot (x-a)^{-1} (x-b)^n$$

$$= (x-a)^m (x-b)^n [n \cdot (x-b)^{-1} + m \cdot (x-a)^{-1}]$$

$$= (x-a)^m (x-b)^n \left(\frac{n}{x-b} + \frac{m}{x-a} \right)$$

$$= (x-a)^n \cdot (x-b)^n \left(\frac{n(x-a) + m(x-b)}{(x-a)(x-b)} \right)$$

$f(x)$ is exist $\forall x$ except at $x=a$ and $x=b \notin (a, b)$

$\therefore f(x)$ is exist in (a, b) .

$\therefore f(x)$ is derivable in (a, b)

$$f(a) = (a-a)^m \cdot (a-b)^n$$

$$= 0 \cdot (a-b)^n$$

$$= 0$$

$$f(b) = (b-a)^m \cdot (b-b)^n$$

$$= (b-a)^m (0)$$

$$= 0$$

$$f(a) = f(b)$$

Then $\exists$ a $c \in (a, b) \ni f'(c) = 0$

$$f(x) = (x-a)^m \cdot (x-b)^n$$

$$f'(x) = (x-a)^m \cdot (x-b)^n \left(\frac{n(x-a) + m(x-b)}{(x-a)(x-b)} \right)$$

$$f'(c) = (c-a)^m \cdot (c-b)^n \left(\frac{n(c-a) + m(c-b)}{(c-a)(c-b)} \right) = 0$$

$$= (c-a)^m (c-b)^n \left[\frac{nc - na + mc - mb}{(c-a)(c-b)} \right] = 0$$

$$= (c-a)^m (c-b)^n \left[\frac{(m+n)c - (na + mb)}{(c-a)(c-b)} \right] = 0$$

$$(c-a)^m = 0, (c-b)^n = 0 \text{ and } (m+n)c - na - mb = 0$$

$$\Rightarrow (m+n)c = na + mb$$

$$c = \frac{na + mb}{m+n} \in (a, b)$$

⑤ $f(x) = \frac{1}{x^2}$ in $[-1, 1]$.

sol $f(x) = \frac{1}{x^2}$

$f(x)$ is ~~not~~ ~~not~~ ~~not~~ exist ~~at~~ ~~at~~ ~~at~~ $x=0$.

$\Rightarrow f(x)$ is ~~not~~ ^{does not} continuous in $[-1, 1]$ except at $x=0$.
 $\in \mathbb{R}$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$\Rightarrow f(x)$ is ~~not~~ ^{does not} derivable in $(-1, 1)$ except at $x=0$.

But $x=0 \in (-1, 1)$

$\therefore$ Rolle's theorem can not applied.

⑥ $f(x) = \sin x$ in $(-\pi, \pi)$.

Soln-

$$f(x) = \sin x$$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $(-\pi, \pi)$.

$$f'(x) = \cos x.$$

$\Rightarrow f'(x)$ is derivable in $(-\pi, \pi)$.

We have to show that $f(-\pi) = f(\pi)$

$$f(-\pi) = \sin(-\pi) = -\sin \pi = 0$$

$$f(\pi) = \sin \pi = 0$$

$$f(-\pi) = f(\pi)$$

then $\exists c \in (-\pi, \pi) \Rightarrow f'(c) = 0$.

$$f(x) = \sin x \Rightarrow f'(x) = \cos x$$

$$f'(c) = 0$$

$$\cos c = 0$$

$$c = \cos^{-1}(0)$$

$$c = \cos^{-1}(\cos \pi/2)$$

$$\boxed{c = \pi/2} \in (-\pi/2, \pi/2)$$

⑦ $f(x) = \tan x$ in $[0, \pi]$.

$$f(x) = \tan x$$

$f(x)$ is exist $\forall x$ except at $x = \pi/2 \in (0, \pi)$.

$\therefore f(x)$ is does not continuous in $[0, \pi]$.

$$f'(x) = \sec^2 x$$

$f'(x)$ ^{is} ~~does not~~ exist $\forall x$ except at $x = 0 \in (0, \pi)$

Rolle's theorem can not be verified.

(8) $f(x) = \sec x$ in $[0, 2\pi]$

$$f(x) = \sec x$$

$f(x)$ is exist $\forall x$ except at $x = \pi/2 \in (0, 2\pi)$

$\Rightarrow f(x)$ is continuous in $[0, 2\pi]$ except at $x = \pi/2 \in (0, 2\pi)$

$$f'(x) = \sec x \cdot \tan x$$

$f'(x)$ is exist $\forall x$ except at $x = \pi/2 \in (0, 2\pi)$

$\Rightarrow f'(x)$ is derivable in $(0, 2\pi)$ except at $x = \pi/2$

$\Rightarrow f(0) = f(2\pi)$ (we have to st)

$$f(0) = \sec 0 = 1$$

$$f(2\pi) = \sec 2\pi = 1$$

$$\boxed{f(0) = f(2\pi)}$$

Then $\forall c \in (0, 2\pi) \ni f'(c) = 0$

$$\sec c \cdot \tan c = 0$$

$$\tan c = 0 \quad \text{and} \quad \sec c = 0$$

$$c = \tan^{-1}(0) \quad c = \sec^{-1}(0)$$

$$c = \tan^{-1}(\tan 0) \quad c = \sec^{-1}(\sec 0)$$

$$\boxed{c = 0}$$

(9) $f(x) = e^x \cdot \sin x$ in $[0, \pi]$

$$f(x) = e^x \sin x$$

$f(x)$ is exist $\forall x$

$\Rightarrow f(x)$ is continuous in $[0, \pi]$

$$f'(x) = e^x \cos x + \sin x e^x$$

$$= e^x (\cos x + \sin x)$$

$f'(x)$ is exist $\forall x$

$\Rightarrow f(x)$ is derivable in $(0, \pi)$

$\Rightarrow$ we have to show that $f(0) = f(\pi)$

$$f(0) = e^0 \cdot \sin 0 = 0$$

$$f(\pi) = e^\pi \sin \pi = 0$$

$$\boxed{f(0) = f(\pi)}$$

Then $\exists a \ c \in (0, \pi) \ni f'(c) = 0$

$$e^c (\cos c + \sin c) = 0$$

$$\cos c + \sin c = 0$$

$$\sin c = -\cos c$$

$$\frac{\sin c}{\cos c} = -1$$

$$\tan c = -1$$

$$c = \tan^{-1}(-1)$$

$$c = \tan^{-1}(-1) \text{ (in } 3\pi/4)$$

$$\boxed{c = 3\pi/4} \in (0, \pi)$$

Saturday

19/10/19

Lagrange's Mean Value Theorem:

Let $f(x)$ be a function of x if

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is derivable in (a, b)

(iii) Then $\exists a \ c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

① $f(x) = x(x-1)(x-2)$ in $[0, 1/2]$

② $f(x) = \log x$ $[1, e]$

③ $f(x) = e^x$ $[0, 1]$

④ $f(x) = \frac{1}{x}$ $[1, 4]$

⑤ $f(x) = x - x^3$ $[-2, 1]$

⑥ If $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$

⑦ By using LMVT, show that $\frac{b-a}{1+b} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a}$ and divides that

⑧ By using LMVT,

$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}(8/5) > \pi/3 - 1/8$

⑨ $x \leq \sin^{-1} x \leq \frac{x}{1-x^2}$

① $f(x) = x(x-1)(x-2)$ in $[0, 1/2]$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, 1/2]$

$$f'(x) = (x^2 - x)(x - 2)$$

$$= x^3 - 2x^2 - x^2 + 2x$$

$$f(x) = x^3 - 3x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, 1/2)$.

$\Rightarrow$ Then $\exists a \in (0, 1/2) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$

$$3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$$

$$\begin{aligned} f(1/2) &= 1/2(1/2-1)(1/2) \\ &= 1/2(-1/2)(1/2) \\ &= -3/8 \end{aligned}$$

$$3c^2 - 6c + 2 = \frac{3}{8} \times \frac{2}{1}$$

$$3c^2 - 6c + 2 - 3/4 = 0$$

$$3c^2 - 6c + 5/4 = 0$$

$$c = \frac{6 \pm \sqrt{36 - 15}}{2(3)}$$

$$= \frac{6 \pm \sqrt{21}}{6}$$

$$= \frac{6}{6} \pm \frac{\sqrt{21}}{6}, \quad \frac{6}{6} - \frac{\sqrt{21}}{6}$$

$$= 1 + \frac{\sqrt{21}}{6}, \quad 1 - \frac{\sqrt{21}}{6}$$

$$\boxed{c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)}$$

② $f(x) = \log x$ in $[1, e]$

$f(x)$ is continuous $\forall x$, except at $x=0 \notin (1, e)$

$\Rightarrow f(x)$ is continuous in $[1, e]$

$$f'(x) = \frac{1}{x}$$

$f'(x)$ is exist $\forall x$, except at $x=0 \notin (1, e)$

$\Rightarrow f'(x)$ is derivable in $(1, e)$.

Then $\exists a \in (1, e) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\frac{1}{c} = \frac{\log e - \log 1}{e - 1}$$

$$\frac{1}{c} = \frac{1 - 0}{e - 1}$$

$$\frac{1}{c} = \frac{1}{e - 1}$$

$$\Rightarrow \boxed{c = e - 1} \in (1, e)$$

$$e = 2.7 \dots$$

$$e - 1 = 2.7 - 1$$

$$= 1.7$$

③ $f(x) = e^x$ in $[0, 1]$

$f(x)$ is continuous $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, 1]$

$f'(x) = e^x$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, 1)$.

Then $\exists a \in (0, 1) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

$$e^c = \frac{e^1 - e^0}{1 - 0}$$

$$e^c = \frac{e - 1}{1}$$

$$e^c = e - 1$$

$$\boxed{c = \log(e - 1) \in (0, 1)}$$

④ $f(x) = \frac{1}{x}$ in $[1, 4]$

$f(x)$ is continuous $\forall x$. except at $x = 0 \notin (1, 4)$

$\Rightarrow f(x)$ is continuous in $[1, 4]$

$f'(x) = -\frac{1}{x^2}$ is exist $\forall x$. except at $x = 0 \notin (1, 4)$

$\Rightarrow f(x)$ is derivable in $(1, 4)$

Then $\exists a \in (1, 4) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

$$-\frac{1}{c^2} = \frac{\frac{1}{4} - 1}{4 - 1}$$

$$-\frac{1}{c^2} = \frac{\frac{1 - 4}{4}}{3}$$

$$-\frac{1}{c^2} = \frac{-\frac{3}{4}}{3}$$

$$\frac{1}{c^2} = \frac{1}{4}$$

$$c^2 = 4$$

$$c = \sqrt{4} \Rightarrow c = \pm 2$$

$$\boxed{c = 2 \in (1, 4)}$$

⑤ $f(x) = x - x^3$ in $[-2, 1]$

$f(x) = x - x^3$ is continuous $\forall x$.

$\Rightarrow f(x)$ is continuous in $[-2, 1]$

$$f'(x) = 1 - 3x^2$$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(-2, 1)$

$$\forall a, c \in (-2, 1) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$1 - 3c^2 = \frac{f(1) - f(-2)}{1 - (-2)}$$

$$1 - 3c^2 = \frac{(1-1) - (-2 - (-8))}{1+2}$$

$$1 - 3c^2 = \frac{0 - (-2 + 8)}{3}$$

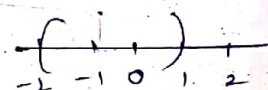
$$3 - 9c^2 = -6$$

$$9c^2 = 9$$

$$c^2 = 1$$

$$c = \pm 1$$

$$\boxed{c = -1 \in (-2, 1)}$$



⑥ If $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$.

Sol:- Let us take $f(x) = \log(1+x)$

Since $f(x) = \log(1+x)$ is continuous $\forall x > 0$.

and $f(x)$ is derivable $\forall x > 0$.

By using L.M.V.T,

$$\forall a, c \in (0, x) \Rightarrow f'(c) = \frac{f(x) - f(0)}{x - 0}$$

We have $f(x) = \log(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - \log(1+0)}{x-0}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - 0}{x}$$

$$\frac{1}{1+c} = \frac{\log(1+x)}{x} \rightarrow \textcircled{1}$$

Given that $0 < c < x$

$$1 < c+1 < x+1$$

$$1 < \frac{1}{c+1} < \frac{1}{x+1}$$

$$1 < \frac{\log(1+x)}{x} < \frac{1}{x+1}$$

$$x < \log(1+x) < \frac{x}{1+x}$$

⑦ Let $f(x) = \tan^{-1}x$ in $[a, b]$

Given, $f(x)$ is continuous ~~if~~ x except
 $f(x)$ is continuous in $[a, b]$
and $f(x)$ is derivable in (a, b) .

By using L.M.V.T,

then $\exists c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b - a}$$

Given that, $a < c < b$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+a^2} > \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b - a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1}(b) - \tan^{-1}(a) > \frac{b-a}{1+b^2}$$

Given that $a=1$, $b=4/3$

$$\frac{4/3-1}{1+(1)^2} > \tan^{-1}(4/3) - \tan^{-1}(1) > \frac{4/3-1}{1+(4/3)^2}$$

$$\frac{1/3}{2} > \tan^{-1}(4/3) - \pi/4 > \frac{1/3}{25/9}$$

$$\frac{1}{6} > \tan^{-1}(4/3) - \pi/4 > \frac{3}{25}$$

$$\frac{1}{6} + \frac{\pi}{4} > \tan^{-1}(4/3) > \frac{3}{25} + \frac{\pi}{4}$$

$$\frac{\pi}{4} + \frac{3}{25} > \tan^{-1}(4/3) > \frac{\pi}{4} + \frac{1}{6}$$

⑧ Let $f(x) = \cos^{-1}x$ in $[a, b]$

Given that, $f(x)$ is continuous in $[a, b]$

and $f(x)$ is derivable in (a, b)

By using L.M.V.T,

$$\exists a < c < b \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = \cos^{-1}x \Rightarrow f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}(b) - \cos^{-1}(a)}{b - a}$$

We know that, $a < c < b$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1 - a^2 > 1 - c^2 > 1 - b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

~~Wrong~~

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\cos^{-1}(a) - \cos^{-1}(b)}{b - a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-a^2}} \geq [\cos^{-1}(a) - \cos^{-1}(b)] \geq \frac{-(b-a)}{\sqrt{1-b^2}}$$

Given that $a=3/5$, $b=1$

$$\frac{a-b}{\sqrt{1-a^2}} > \cos^{-1}(b) - \cos^{-1}(a) > \frac{a-b}{\sqrt{1-b^2}}$$

Given that

Monday
21/10 Cauchy's Mean Value Theorem

Let $f(x), g(x)$ are functions of 'x'.

(i) $f(x), g(x)$ are continuous in $[a, b]$

(ii) $f(x), g(x)$ are derivable in (a, b)

$$\text{Then } \exists c \in (a, b) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Verify the Cauchy's Mean Value Theorem for the following functions.

① $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$ $0 < a < b$.

② $f(x) = \sin x$, $g(x) = \cos x$ in $[0, \pi/2]$

③ $f(x) = e^x$, $g(x) = e^{-x}$ in $[a, b]$

④ $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a, b]$ if $0 < a < b$

⑤ $f(x) = x^2 + 2$, $g(x) = x^3 - 1$ in $[1, 2]$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1, e]$

⑦ $f(x) = x^3$, $g(x) = 2 - x$ in $[0, 9]$

① $f(x)$ is always continuous $\forall x$.

$g(x)$ is continuous $\forall x$ except at $x=0 \notin (a, b)$ $(0 < a < b)$

$\Rightarrow f(x), g(x)$ are continuous in $[a, b]$.

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad g'(x) = -\frac{1}{2} x^{-3/2}$$

$f'(x)$ is exist $\forall x$ except at $x=0 \notin (a, b)$

$f(x)$ is derivable in (a, b) .

$g'(x)$ is exist $\forall x$ except at $x=0 \in (a, b)$

$g(x)$ is derivable in (a, b)

$\Rightarrow f(x), g(x)$ are derivable in (a, b)

$$\text{Then } \exists c \in (a, b) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2\sqrt{c}^3}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$-c = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$+c = \frac{\sqrt{b}-\sqrt{a}}{+(\sqrt{b}-\sqrt{a})} \sqrt{ab}$$

$$\boxed{c = \sqrt{ab} \in (a, b)}$$

② $f(x) = \sin x$, $g(x) = \cos x$ $[0, \pi/2]$

$\Rightarrow f(x), g(x)$ are always continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0, \pi/2]$.

$$f'(x) = \cos x, \quad g'(x) = -\sin x$$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in $(0, \pi/2)$.

$$\text{Then } \exists a \in (0, \pi/2) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin \pi/2 - \sin 0}{\cos \pi/2 - \cos 0}$$

$$\frac{\cos c}{-\sin c} = \frac{1-0}{0-1}$$

$$\frac{\cos c}{-\sin c} = \frac{1}{-1}$$

$$\cos c = \sin c$$

$$\frac{\sin c}{\cos c} = 1$$

$$\tan c = 1$$

$$c = \tan^{-1}(1)$$

$$\boxed{c = \pi/4 \in (0, \pi/2)}$$

③ $f(x) = e^x$, $g(x) = e^{-x}$ in $[a, b]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[a, b]$.

$$f(x) = e^x$$

$$g(x) = e^{-x}$$

$$f'(x) = e^x$$

$$g'(x) = -e^{-x}$$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in (a, b)

$$\text{Then } \exists c \in (a, b) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$-e^c \cdot e^c = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-(e^c)^2 = \frac{e^b - e^a}{\frac{e^a - e^b}{e^a e^b}}$$

$$+e^{2c} = \frac{e^b - e^a}{+ (e^b - e^a)} e^a e^b$$

$$e^{2c} = e^a e^b$$

$$e^{2c} = e^{a+b}$$

$$2c = a + b$$

$$\boxed{c = \frac{a+b}{2} \in (a, b)}$$

⑤ $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a, b]$ ($0 < a < b$)

$f(x)$ is continuous $\forall x$, except at $x=0 \notin (a, b)$

$g(x)$ is continuous $\forall x$, except at $x=0 \notin (a, b)$

$\Rightarrow f(x), g(x)$ are continuous in $[a, b]$

$$f'(x) = -2x^{-3}, \quad g'(x) = \log x \cdot \frac{1}{x^2}$$

$f'(x)$ is exist $\forall x$, except at $x=0$.

$g'(x)$ is exist $\forall x$, except at $x=0$

$\Rightarrow f(x), g(x)$ are derivable in (a, b)

$$\text{Then } \exists c \in (a, b) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{-\frac{2}{c^3}}{\frac{1}{c^2}} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{x} = \frac{\frac{a^2-b^2}{ab}}{\frac{a-b}{ab}}$$

$$\frac{2}{x} = \frac{(a+b)(a-b)}{(ab)^2} \times \frac{ab}{a-b}$$

$$\boxed{c = \frac{2ab}{a+b} \in (a,b)}$$

⑤ $f(x) = x^2 + 2$, $g(x) = x^3 - 1$, in $[1, 2]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[1, 2]$.

$$f'(x) = 2x, \quad g'(x) = 3x^2$$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in $(1, 2)$

$$\text{Then } \exists a \in (1, 2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{2c}{3c^2} = \frac{[(2)^2 + 2] - [(1)^2 + 2]}{[(2)^3 - 1] - [(1)^3 - 1]}$$

$$\frac{2}{3c} = \frac{(4+2) - (1+2)}{(8-1) - (1-1)}$$

$$\frac{2}{3c} = \frac{6-3}{7-0}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$\boxed{c = \frac{14}{9} \in (1, 2)}$$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1, e]$

$f(x)$ is continuous $\forall x$, except at $x=0 \notin (1, e)$

$g(x)$ is continuous $\forall x$, except at $x=0 \notin (1, e)$

$\Rightarrow f(x), g(x)$ are continuous in $[1, e]$

$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{x^2}$$

$f'(x)$ is exist $\forall x$ except at $x=0 \notin (1,e)$

$g'(x)$ is exist $\forall x$ except at $x=0 \notin (1,e)$

$\Rightarrow f(x), g(x)$ is derivable in $(1,e)$.

$$\text{Then } \exists c \in (1,e) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{e}}{\frac{1}{e^2}} = \frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}}$$

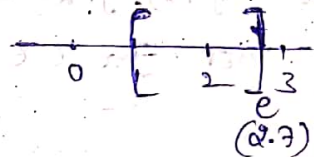
$$-c = \frac{\log_e e - 0}{\frac{1-e}{e}}$$

$$-c = \frac{1-0}{\frac{1-e}{e}}$$

$$-c = \frac{e}{1-e}$$

$$c = \frac{e}{e-1} \in (1,e)$$

$$c = 1.58$$



⑦ $f(x) = x^3, \quad g(x) = 2-x$ in $[0,9]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0,9]$

$$f'(x) = 3x^2$$

$f'(x)$ is exist $\forall x$.

$f(x)$ is derivable in $(0,9)$

$$g'(x) = 0-1 = -1$$

$g'(x)$ is exist $\forall x$.

$g(x)$ is derivable in $(0,9)$.

$\Rightarrow f(x), g(x)$ are derivable in $(0,9)$

Then $\exists c \in (0, 9) \cdot \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{3c^2}{-1} = \frac{(9)^3 - (0)^3}{(2-9) - (2-0)}$$

$$+ 3c^2 = \frac{81 \times 9 - 8}{-7 - 2}$$

$$c^2 = \frac{81 \times 9}{7}$$

$$-3c^2 = \frac{721}{-7-2}$$

$$+3c^2 = \frac{721}{+9}$$

$$c^2 = \frac{721}{27}$$

$$c = \sqrt{\frac{721}{27}}$$

$$c = 5.1675 \in (0, 9)$$

Wednesday

23/10 Taylor's Expansion And Maclaurin's:

Taylor's expansion at $x=a$, $x=1$, $x=\pi/2$.

Taylor's expansion about $x=a$
at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

This is also called as Taylor's expansion in powers of $(x-a)$.

Maclaurin's:

at $x=0$, $f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$

① $f(x) = \sin x$

② $f(x) = \log(1+x)$

③ $f(x) = \tan^{-1} x$

④ $f(x) = e^x$ at $x=1$

⑤ $f(x) = (1-x)^{5/2}$

⑥ $f(x) = \log x$ in powers of $x-1$ and hence evaluate $\log 1.1$ correct to four decimal process.

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x=2$.

④ $f(x) = \log x$

By Taylor's expansion at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \Rightarrow f^{(4)}(1) = -6$$

$$\therefore f(x) = \log x = \log 1 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

⑤ $f(x) = \log(x)$

By Taylor's expansion at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

at $a=1$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \dots$$

$$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \Rightarrow f^{(4)}(1) = -6$$

from ①,

$$\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \dots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \dots$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \dots$$

$$= 0.1 - 0.005 + 0.0003 - 0.000025$$

No (1)0

$$= \underline{\underline{0.105202}}$$

$$\log(1.1) = 0.095310179$$

$$\boxed{\log(1.1) \approx 0.094}$$

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x=2$.

By Taylor's expansion at $x=2$

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \frac{(x-2)^4}{4!}f^{(4)}(2) + \dots \rightarrow \textcircled{1}$$

$$f(x) = 2x^3 - 7x^2 + x + 6 \Rightarrow f(2) = -4$$

$$f'(x) = 6x^2 - 14x + 1 \Rightarrow f'(2) = 24 - 28 + 1 = -3$$

$$f''(x) = 12x - 14 \Rightarrow f''(2) = 24 - 14 = 10$$

$$f'''(x) = 12 \Rightarrow f'''(2) = 12$$

$$f^{(4)}(x) = 0 \Rightarrow f^{(4)}(2) = 0$$

from ①,

$$2x^3 - 7x^2 + x + 6 = -4 + (x-2) \frac{(-3)}{1!} + \frac{(x-2)^2}{2!} 10 + \frac{(x-2)^3}{3!} 12 + \dots$$

$$= -4 + (x-2) \frac{(-3)}{1!} + \frac{(x-2)^2}{2!} 10 + \frac{(x-2)^3}{3!} 12$$

$$= -4 - 3(x-2) + 5(x-2)^2 + 2(x-2)^3$$

② $f(x) = \log(1+x)$

Now, the Maclaurin's expansion is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \rightarrow \textcircled{1}$$

$$f(x) = \log(1+x) \Rightarrow f(0) = \log(1+0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{(4)}(0) = -6$$

from ①,

$$\begin{aligned} \log(1+x) &= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots \\ &= x - \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} - 6 \cdot \frac{x^4}{4!} + \dots \end{aligned}$$

⑤ $f(x) = (1-x)^{5/2}$

By Maclaurin's expansion is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots \rightarrow \textcircled{1}$$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$$

$$f'(x) = -\frac{5}{2} (1-x)^{3/2} \Rightarrow f'(0) = -\frac{5}{2}$$

$$f''(x) = -\frac{5}{2} \cdot \frac{3}{2} (1-x)^{1/2} \Rightarrow f''(0) = -\frac{15}{4}$$

$$f'''(x) = \frac{15}{4} \cdot \frac{1}{2} (1-x)^{-1/2} \Rightarrow f'''(0) = \frac{15}{8}$$

from ①,

$$(1-x)^{5/2} = 1 + x \cdot \left(-\frac{5}{2}\right) + \frac{x^2}{2!} \left(-\frac{15}{4}\right) + \frac{x^3}{3!} \left(\frac{15}{8}\right) + \dots$$

$$= 1 - \frac{5}{2}x + \frac{15x^2}{4 \cdot 2!} - \frac{15}{8} \cdot \frac{x^3}{3!} + \dots$$

$$= 1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{5}{16}x^3 + \dots$$

① $f(x) = \sin x$.

Now, the Maclaurin's expansion is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots$$

$$f(x) = \sin x \Rightarrow f(0) = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = -(-\sin x) \Rightarrow f^{(4)}(0) = 0$$

from ①,

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \dots$$

② $f(x) = \tan^{-1} x$.

Now, Maclaurin's expansion is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots$$

$$f(x) = \tan^{-1} x \Rightarrow f(0) = \tan^{-1}(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x^2)^2} (2x) \Rightarrow f''(0) = 0$$

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = \frac{-2(1+0)^2 + 0}{(1+0)^4} = -2$$

from ①,

$$\tan^{-1} x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \dots$$

$$\tan^{-1} x = x - 2 \frac{x^3}{3!} + \dots$$

④ $f(x) = e^x$ at $x=1$

Now, Taylor's expansion ~~is~~ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \dots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

$\Rightarrow$ at $a=1$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \dots \rightarrow \textcircled{1}$$

$$f(x) = e^x \Rightarrow f(1) = e$$

$$f'(x) = e^x \Rightarrow f'(1) = e$$

$$f''(x) = e^x \Rightarrow f''(1) = e$$

$$f'''(x) = e^x \Rightarrow f'''(1) = e$$

$$f^{IV}(x) = e^x \Rightarrow f^{IV}(1) = e$$

from $\textcircled{1}$,
 $\textcircled{2}$

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \dots$$

$$e^x = e \left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots \right]$$

16/11/19
Saturday

Unit - 4

Partial Derivatives

1. Homogeneous function, Euler's Theorem, Total derivatives, chain rule, Jacobian, Functionally dependents;
2. Taylor's and Maclaurin's expansions with two variables.

Applications: Maxima and minima with constants and without constants, Lagranges

(I)

③ If $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ (or) ~~Prove~~ prove that $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \sin 2U$.

Sol:- Given $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\tan U = \frac{x^3+y^3}{x+y}$$

$$\tan U = \frac{x^2 \left(1 + \frac{y^3}{x^3}\right)}{x \left(1 + \frac{y}{x}\right)}$$

$$\tan U = x^2 \left[\frac{1 + \left(\frac{y}{x}\right)^3}{1 + \frac{y}{x}} \right]$$

$$\tan U = x^2 \cdot f\left(\frac{y}{x}\right)$$

→ $\tan U$ is homogeneous of degree 2.

By Euler's theorem,

$$x \cdot \frac{\partial \tan U}{\partial x} + y \cdot \frac{\partial \tan U}{\partial y} = 2 \cdot \tan U$$

$$x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} + y \cdot \sec^2 U \cdot \frac{\partial U}{\partial y} = 2 \cdot \tan U$$

$$\sec^2 U \left(x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} \right) = 2 \cdot \tan U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{2 \cdot \tan U}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2 \cdot \frac{\sin U}{\cos U} \times \cos^2 U$$

$$\boxed{x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \sin 2U}$$

④ If $u = \sin^{-1} \left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} \right)$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -3 \tan u$

Solr

Given $u = \sin^{-1} \left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} \right)$

$$u = \sin^{-1} \left(\frac{x \left(1 + 2 \frac{y}{x} + 3 \frac{z}{x} \right)}{x^4 \sqrt{1 + \frac{y^8}{x^8} + \frac{z^8}{x^8}}} \right)$$

$$\sin u = x^{-3} \left[\frac{1 + 2 \frac{y}{x} + 3 \left(\frac{z}{x} \right)}{\sqrt{1 + \left(\frac{y}{x} \right)^8 + \left(\frac{z}{x} \right)^8}} \right]$$

$$\sin u = x^{-3} \cdot f \left(\frac{y}{x}, \frac{z}{x} \right)$$

$\therefore \sin u$ is homogeneous of degree -3 .

By Euler's theorem,

$$x \cdot \frac{\partial \sin u}{\partial x} + y \cdot \frac{\partial \sin u}{\partial y} + z \cdot \frac{\partial \sin u}{\partial z} = -3 \sin u$$

$$x \cdot \cos u \cdot \frac{\partial u}{\partial x} + y \cdot \cos u \cdot \frac{\partial u}{\partial y} + z \cdot \cos u \cdot \frac{\partial u}{\partial z} = -3 \sin u$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right) = -3 \sin u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = -3 \frac{\sin u}{\cos u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = -3 \tan u.$$

⑤ $u = \log \left(\frac{x^4+y^4}{x+y} \right)$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$.

Solr

Given $u = \log \left(\frac{x^4+y^4}{x+y} \right)$

$$e^u = \frac{x^4 \left(1 + \frac{y^4}{x^4} \right)}{x \left(1 + \frac{y}{x} \right)}$$

$$e^u = x^3 \left(\frac{1 + \left(\frac{y}{x} \right)^4}{1 + \frac{y}{x}} \right)$$

$$e^u = x^3 \cdot f \left(\frac{y}{x} \right)$$

$\therefore e^u$ is homogeneous of degree 3 .

By Euler's Theorem,

$$x \cdot \frac{\partial e^u}{\partial x} + y \cdot \frac{\partial e^u}{\partial y} = 3 \cdot e^u$$

$$x \cdot e^y \frac{\partial u}{\partial x} + y e^y \frac{\partial u}{\partial y} = 3 \cdot e^y$$

$$e^y \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 3 \cdot e^y$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3}$$

⑦ $U = x f\left(\frac{y}{x}\right)$ prove that $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = U$.

Given $U = x f\left(\frac{y}{x}\right)$

$\therefore U$ is the homogeneous of degree "1".

By Euler's theorem,

$$x \cdot \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = n \cdot U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = (1) U$$

$$\boxed{x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = U}$$

① $U = (x^{1/2} + y^{1/2})(x^n + y^n)$ verify the Euler's theorem.

Given $U = (x^{1/2} + y^{1/2})(x^n + y^n)$

$$U = x^{1/2} \left(1 + \frac{y^{1/2}}{x^{1/2}} \right) x^n \left(1 + \frac{y^n}{x^n} \right)$$

$$= x^{n+1/2} \left[\left(1 + \left(\frac{y}{x} \right)^{1/2} \right) \left(1 + \left(\frac{y}{x} \right)^n \right) \right]$$

$$U = x^{n+1/2} f\left(\frac{y}{x}\right)$$

$\therefore U$ is the homogeneous of degree " $n + \frac{1}{2}$ ".

By Euler's theorem,

$$x \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n \cdot U$$

$$\boxed{x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \left(n + \frac{1}{2}\right) U}$$

We have to prove that $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \left(n + \frac{1}{2}\right) U$.

$$\frac{\partial}{\partial x} (U) = \frac{\partial}{\partial x} \left[(x^{1/2} + y^{1/2})(x^n + y^n) \right]$$

$$= (x^{1/2} + y^{1/2})(n x^{n-1} + 0) + (x^n + y^n) \left(\frac{1}{2} x^{-1/2} + 0 \right)$$

$$= (x^{1/2} + y^{1/2}) n x^{n-1} + (x^n + y^n) \frac{1}{2} x^{-1/2}$$

$$x \cdot \frac{\partial U}{\partial x} = n \cdot x^n (x^{1/2} + y^{1/2}) + \frac{1}{2} x^{1/2} (x^n + y^n)$$

Similarly, $\frac{\partial U}{\partial y} = n \cdot y^{n-1} (x^{1/2} + y^{1/2}) + \frac{1}{2} y^{-1/2} (x^n + y^n)$

$$y \frac{\partial U}{\partial y} = n \cdot y^n (x^{1/2} + y^{1/2}) + \frac{1}{2} y^{1/2} (x^n + y^n)$$

L.H.S

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y}$$

$$= n \cdot x^n (x^{1/2} + y^{1/2}) + \frac{1}{2} x^{1/2} (x^n + y^n) + n y^n (x^{1/2} + y^{1/2}) + \frac{1}{2} y^{1/2} (x^n + y^n)$$

$$= n(x^{1/2} + y^{1/2}) (x^n + y^n) + \frac{1}{2} (x^n + y^n) (x^{1/2} + y^{1/2})$$

$$= (x^n + y^n) (x^{1/2} + y^{1/2}) (n + \frac{1}{2})$$

$$= (n + \frac{1}{2}) U$$

$$= R.H.S$$

$\therefore$ Euler's theorem verified.

② $U = \sin^{-1}(\frac{y}{x}) + \tan^{-1}(\frac{y}{x})$. Verify the Euler's theorem.

Given $U = \sin^{-1}(\frac{y}{x}) + \tan^{-1}(\frac{y}{x})$

$$= \csc^{-1}(\frac{y}{x}) + \tan^{-1}(\frac{y}{x})$$

$$U = x^0 [\csc^{-1}(\frac{y}{x}) + \tan^{-1}(\frac{y}{x})]$$

$$U = x^0 f(\frac{y}{x})$$

$\therefore U$ is homogeneous of degree "0".

By Euler's theorem,

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = n \cdot U$$

$$= (0) U = 0.$$

We have to prove that $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = 0$.

Ans

$$\frac{\partial}{\partial x} (U) = \frac{\partial}{\partial x} \left[\sin^{-1}(\frac{y}{x}) + \tan^{-1}(\frac{y}{x}) \right]$$

$$= \frac{1}{\sqrt{1 - (\frac{y}{x})^2}} \left(-\frac{y}{x^2}\right) + \frac{1}{1 + (\frac{y}{x})^2} \left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2} \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} + -\frac{y}{x^2} \frac{1}{\frac{x^2 + y^2}{x^2}}$$

$$= -\frac{1}{y \sqrt{1 - \frac{y^2}{x^2}}} + -\frac{y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\Rightarrow x \cdot \frac{\partial v}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$\frac{\partial v}{\partial y} (v) = \frac{d}{dy} \left[\sin^{-1} \left(\frac{x}{y} \right) + \tan^{-1} \left(\frac{y}{x} \right) \right]$$

$$= \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot x \left(-\frac{1}{y^2} \right) + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{-x}{y \sqrt{y^2 - x^2}} + \frac{1}{x \cdot \left(\frac{x^2 + y^2}{x^2} \right)}$$

$$\frac{\partial v}{\partial y} = \frac{-x}{y \sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$\Rightarrow y \frac{\partial v}{\partial y} = \frac{-xy}{y \sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$$= \frac{-x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

L.H.S

$$x \cdot \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y}$$

$$= \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} - \frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$$= 0$$

$$= \underline{\text{R.H.S}}$$

$\therefore$ Euler's theorem verified.

⑥ $v = \log \left(\frac{x^2 + y^2}{xy} \right)$ verify the Euler's theorem.

Sol:-

$$\text{Given } v = \log \left(\frac{x^2 + y^2}{xy} \right)$$

$$e^v = \frac{x^2 \left(1 + \frac{y^2}{x^2} \right)}{xy}$$

$$e^v = \frac{x \left(1 + \left(\frac{y}{x} \right)^2 \right)}{x \cdot (y/x)}$$

$$e^v = x^0 f \left(\frac{y}{x} \right)$$

$\therefore e^v$ is homogeneous of degree '0'.

By Euler's theorem, $x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = n \cdot v$
 $= (0) v = \underline{0}$

We have to prove that, $x \frac{du}{dx} + y \frac{du}{dy} = 0$.

$$\begin{aligned}
 \frac{d}{dx}(u) &= \frac{d}{dx} \left(\log \left(\frac{x^2+y^2}{xy} \right) \right) \\
 &= \frac{1}{\frac{x^2+y^2}{xy}} \left[\frac{xy(2x+0) - (x^2+y^2) \cdot y}{(xy)^2} \right] \\
 &= \frac{xy}{x^2+y^2} \left[\frac{xy(2x) - (x^2+y^2)y}{(xy)^2} \right] \\
 &= \frac{1}{x^2+y^2} \left[\frac{2x^2y - x^2y - y^3}{xy} \right] \\
 &= \frac{1}{x^2+y^2} \left[\frac{x^2y - y^3}{xy} \right] \\
 &= \frac{1}{x^2+y^2} \cdot y \left(\frac{x^2-y^2}{xy} \right) \\
 \frac{du}{dx} &= \frac{x^2-y^2}{x(x^2+y^2)}
 \end{aligned}$$

$$\Rightarrow x \cdot \frac{du}{dx} = \frac{x \cdot (x^2-y^2)}{x(x^2+y^2)} = \frac{x^2-y^2}{x^2+y^2}$$

$$\begin{aligned}
 \frac{d}{dy}(u) &= \frac{d}{dy} \left(\log \left(\frac{x^2+y^2}{xy} \right) \right) \\
 &= \frac{1}{\frac{x^2+y^2}{xy}} \left[\frac{xy(0+2y) - (x^2+y^2)x}{(xy)^2} \right] \\
 &= \frac{1}{x^2+y^2} \left[\frac{2xy^2 - x^3 - xy^2}{xy} \right] \\
 &= \frac{1}{x^2+y^2} \left(\frac{xy^2 - x^3}{xy} \right) \\
 &= \frac{1}{x^2+y^2} \cdot x \left(\frac{y^2-x^2}{xy} \right) \\
 \frac{du}{dy} &= \frac{y^2-x^2}{y(x^2+y^2)}
 \end{aligned}$$

$$\Rightarrow y \cdot \frac{du}{dy} = (y) \cdot \frac{y^2-x^2}{y(x^2+y^2)} = \frac{y^2-x^2}{x^2+y^2}$$

$$\begin{aligned}
 \text{L.H.S} \quad x \frac{du}{dx} + y \frac{du}{dy} &= \frac{x^2-y^2}{x^2+y^2} + \frac{y^2-x^2}{x^2+y^2} \\
 &= \frac{x^2-y^2+y^2-x^2}{x^2+y^2}
 \end{aligned}$$

$$= \frac{0}{x^2+y^2} = 0 = \text{R.H.S}$$

$\therefore$ Euler's theorem verified.

⑧ $U = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ - Verify the Euler's theorem.

Solr Given $U = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

$$U = \frac{x^{1/4} \left[1 + \frac{y^{1/4}}{x^{1/4}} \right]}{x^{1/5} \left[1 + \frac{y^{1/5}}{x^{1/5}} \right]}$$

$$U = x^{1/4} \cdot x^{-1/5} \left[\frac{1 + (y/x)^{1/4}}{1 + (y/x)^{1/5}} \right]$$

$$U = x^{1/4 - 1/5} \left[\frac{1 + (y/x)^{1/4}}{1 + (y/x)^{1/5}} \right]$$

$$U = x^{\frac{5-4}{20}} \left[\frac{1 + (y/x)^{1/4}}{1 + (y/x)^{1/5}} \right]$$

$$U = x^{1/20} f\left(\frac{y}{x}\right)$$

$\therefore U$ is homogeneous of degree $\frac{1}{20}$.

By Euler's theorem, $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = n \cdot U$

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \frac{1}{20} U$$

We have to prove that,

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \frac{1}{20} U$$

$$\frac{\partial}{\partial x} (U) = \frac{\partial}{\partial x} \left(\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} \right)$$

$$= \frac{(x^{1/5} + y^{1/5}) \left(\frac{1}{4} x^{-3/4} + 0 \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} x^{-4/5} + 0 \right)}{(x^{1/5} + y^{1/5})^2}$$

$$\frac{\partial U}{\partial x} = \frac{\frac{1}{4} x^{-3/4} (x^{1/5} + y^{1/5}) - \frac{1}{5} x^{-4/5} (x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$\Rightarrow x \frac{\partial U}{\partial x} = \frac{\frac{1}{4} x^{-3/4+1} (x^{1/5} + y^{1/5}) - \frac{1}{5} x^{-4/5+1} (x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{\frac{1}{4} x^{1/4} (x^{1/5} + y^{1/5}) - \frac{1}{5} x^{1/5} (x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$\frac{d}{dy}(v) = \frac{d}{dy} \left(\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} \right)$$

$$= \frac{(x^{1/5} + y^{1/5})(0 + \frac{1}{4}y^{-3/4}) - (x^{1/4} + y^{1/4})(0 + \frac{1}{5}y^{-4/5})}{(x^{1/5} + y^{1/5})^2}$$

$$\frac{dv}{dy} = \frac{\frac{1}{4}y^{-3/4}(x^{1/5} + y^{1/5}) - \frac{1}{5}y^{-4/5}(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$\Rightarrow y \frac{dv}{dy} = \frac{\frac{1}{4}y^{-3/4+1}(x^{1/5} + y^{1/5}) - \frac{1}{5}y^{-4/5+1}(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{\frac{1}{4}y^{1/4}(x^{1/5} + y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

L.H.S

$$x \cdot \frac{du}{dx} + y \frac{dv}{dy}$$

$$= \frac{\frac{1}{4}x^{1/4}(x^{1/5} + y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2} + \frac{\frac{1}{4}y^{1/4}(x^{1/5} + y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{\frac{1}{4}x^{1/4}(x^{1/5} + y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4} + y^{1/4}) + \frac{1}{4}y^{1/4}(x^{1/5} + y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{\frac{1}{4}(x^{1/5} + y^{1/5})[x^{1/4} + y^{1/4}] - \frac{1}{5}(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})(\frac{1}{4} - \frac{1}{5})}{(x^{1/5} + y^{1/5})^2} = \frac{(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})(\frac{1}{20})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{1}{20} \left(\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} \right)$$

$$= \frac{1}{20} v$$

$$= R.H.S$$

∴ Euler's theorem verified.

22/11/2019
Friday (II)

Q) If $U = \frac{x^2y}{x+y}$ show that $x \frac{\partial^2 U}{\partial x^2} + y \frac{\partial^2 U}{\partial y \partial x} = 2 \frac{\partial U}{\partial x}$.

Soln

Given $U = \frac{x^2y}{x+y}$

$$U = \frac{x^2y}{x(1+\frac{y}{x})} = x^2 \left(\frac{\frac{y}{x}}{1+\frac{y}{x}} \right)$$

$$U = x^2 \left(\frac{y/x}{1+y/x} \right)$$

$\therefore U$ is homogeneous of degree "2".

By Euler's theorem $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = 2U$.

diff. w. x to 'x' partially

$$(1) \frac{\partial U}{\partial x} + x \frac{\partial^2 U}{\partial x^2} + (2) \frac{\partial U}{\partial y} + y \frac{\partial^2 U}{\partial x \partial y} = 2 \frac{\partial U}{\partial x}$$

$$\frac{\partial U}{\partial x} + x \frac{\partial^2 U}{\partial x^2} + y \frac{\partial^2 U}{\partial x \partial y} = 2 \frac{\partial U}{\partial x}$$

$$x \frac{\partial^2 U}{\partial x^2} + y \frac{\partial^2 U}{\partial x \partial y} = 2 \frac{\partial U}{\partial x} - \frac{\partial U}{\partial x}$$

$$\boxed{x \frac{\partial^2 U}{\partial x^2} + y \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial U}{\partial x}}$$

Q) If $U = \tan^{-1} \left(\frac{x^3+y^3}{x+y} \right)$ prove that $x^2 \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \frac{\partial^2 U}{\partial y^2} = 3 \sin 4U - \sin 2U = 2 \cos 3U \sin U$

Soln

Given $U = \tan^{-1} \left(\frac{x^3+y^3}{x+y} \right)$

$$\tan U = \frac{x^3 \left[1 + \left(\frac{y}{x} \right)^3 \right]}{x \left(1 + \frac{y}{x} \right)}$$

$$\tan U = x^2 \left[\frac{1 + (y/x)^3}{1 + (y/x)} \right]$$

$\therefore \tan U$ is homogeneous of degree "2".

By Euler's theorem, $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = n \cdot U$

$$x \frac{\partial}{\partial x} (\tan U) + y \frac{\partial}{\partial y} (\tan U) = 2 \tan U \rightarrow (1)$$

$$x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} + y \cdot \sec^2 U \cdot \frac{\partial U}{\partial y} = 2 \tan U$$

$$\sec^2 u \left[x \frac{du}{dx} + y \frac{du}{dy} \right] = 2 \tan u$$

$$x \frac{du}{dx} + y \frac{du}{dy} = 2 \cdot \frac{\sin u}{\cos u} \times \cos^2 u$$

$$x \frac{du}{dx} + y \frac{du}{dy} = \sin 2u \rightarrow (2)$$

diff. w. r. to 'x' partially.

$$(1) \frac{du}{dx} + x \cdot \frac{d^2u}{dx^2} + y \cdot \frac{d^2u}{dx dy} = \cos 2u \quad (2) \frac{du}{dx}$$

$$\frac{du}{dx} + x \frac{d^2u}{dx^2} + y \frac{d^2u}{dx dy} = 2 \cos 2u \frac{du}{dx}$$

$$x \frac{d^2u}{dx^2} + y \frac{d^2u}{dx dy} = 2 \cos 2u \cdot \frac{du}{dx} + \frac{du}{dx}$$

$$x \frac{d^2u}{dx^2} + y \frac{d^2u}{dx dy} = (2 \cos 2u - 1) \frac{du}{dx}$$

$$x \frac{du}{dx} + x^2 \frac{d^2u}{dx^2} + xy \frac{d^2u}{dx dy} = 2 \cos 2u \cdot x \frac{du}{dx} \rightarrow (3)$$

from (2),

$$\text{Ily, } y \cdot \frac{du}{dy} + y^2 \frac{d^2u}{dy^2} + xy \frac{d^2u}{dy dx} = 2 \cos u \cdot y \cdot \frac{du}{dy} \rightarrow (4)$$

Adding (3) & (4)

$$x \cdot \frac{du}{dx} + y \frac{du}{dy} + x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = 2 \cos u \left[x \frac{du}{dx} + y \frac{du}{dy} \right]$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = 2 \cos 2u \left(x \frac{du}{dx} + y \frac{du}{dy} \right) - \left(x \frac{du}{dx} + y \frac{du}{dy} \right)$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = 2 \cos 2u \sin 2u - \sin 2u$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = \sin 2u (2 \cos 2u - 1)$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = 2 \cos \left(\frac{4u+2u}{2} \right) \cdot \sin \left(\frac{4u-2u}{2} \right)$$

$$x^2 \frac{d^2u}{dx^2} + 2xy \frac{d^2u}{dx dy} + y^2 \frac{d^2u}{dy^2} = 2 \cos 3u \sin u$$

(17) If $u = \tan^{-1} \left(\frac{y^2}{x} \right)$ show that $x^2 \frac{d^2u}{dx^2} + 2xy \frac{d^2u}{dx dy} + y^2 \frac{d^2u}{dy^2} = -\sin 2u \cdot \sin^2 u$

Soln Given $u = \tan^{-1} \left(\frac{y^2}{x} \right)$

$$\tan u = \frac{y^2}{x}$$

$$\tan u = \frac{xy^2}{x^2}$$

$$\tan u = x \left(\frac{y}{x} \right)^2 \Rightarrow \tan u = x \cdot f\left(\frac{y}{x}\right)$$

$\therefore \tan u$ is homogeneous of degree '1'.

By Euler's theorem, $x \frac{du}{dx} + y \frac{du}{dy} = nu$.

$$x \cdot \frac{d}{dx}(\tan u) + y \cdot \frac{d}{dy}(\tan u) = \tan u. \rightarrow (1)$$

$$x \cdot \sec^2 u \cdot \frac{du}{dx} + y \sec^2 u \cdot \frac{du}{dy} = \tan u.$$

$$\sec^2 u \left(x \frac{du}{dx} + y \frac{du}{dy} \right) = \tan u$$

$$x \frac{du}{dx} + y \frac{du}{dy} = \frac{\sin u}{\cos u} \times \cos^2 u$$

$$x \cdot \frac{du}{dx} + y \frac{du}{dy} = \sin u \cdot \cos u.$$

$$x \cdot \frac{du}{dx} + y \frac{du}{dy} = \frac{1}{2} \sin 2u. \rightarrow (2)$$

diff. w. r. to "x" partially

$$(1) \frac{du}{dx} + x \cdot \frac{d^2 u}{dx^2} + y \frac{d^2 u}{dx dy} = \frac{1}{2} \cos 2u \cdot \frac{du}{dx}$$

$$\frac{du}{dx} + x \cdot \frac{d^2 u}{dx^2} + y \frac{d^2 u}{dx dy} = \cos 2u \cdot \frac{du}{dx}$$

$$x \cdot \frac{du}{dx} + x^2 \frac{d^2 u}{dx^2} + xy \frac{d^2 u}{dx dy} = x \cos 2u \cdot \frac{du}{dx} \rightarrow (3)$$

from (2),

$$\text{Ily, } y \frac{du}{dy} + y^2 \frac{d^2 u}{dy^2} + xy \frac{d^2 u}{dx dy} = y \cos 2u \cdot \frac{du}{dx} \rightarrow (4)$$

(3) + (4) $\Rightarrow$

$$x \cdot \frac{du}{dx} + y \frac{du}{dy} + x^2 \frac{d^2 u}{dx^2} + y^2 \frac{d^2 u}{dy^2} + 2xy \frac{d^2 u}{dx dy} = \cos 2u (x \frac{du}{dx} + y \frac{du}{dy})$$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx dy} + y^2 \frac{d^2 u}{dy^2} = x \frac{du}{dx} + y \frac{du}{dy} (\cos 2u - 1)$$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx dy} + y^2 \frac{d^2 u}{dy^2} = \frac{1}{2} \sin 2u (\cos 2u - 1)$$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx dy} + y^2 \frac{d^2 u}{dy^2} = \frac{1}{2} \sin 2u (1 - \cos 2u - 1)$$

$$\frac{1}{2} \sin 2u (1 - 2 \sin^2 u - 1)$$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx dy} + y^2 \frac{d^2 u}{dy^2} = -\frac{1}{2} \sin 2u \sin^2 u$$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx dy} + y^2 \frac{d^2 u}{dy^2} = -\sin 2u \cdot \sin^2 u.$$

* If $U = (x^2 + y^2)^{1/3}$. Show that $x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = -\frac{2U}{9}$.

* If $U = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$. Then evaluate $x^2 \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \frac{\partial^2 U}{\partial y^2}$.

* If $U = \operatorname{cosec}^{-1}\left(\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$. Evaluate $x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y}$.

(10) If $U = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ Prove that $x^2 \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \frac{\partial^2 U}{\partial y^2} = -\frac{\sin U \cos 2U}{4 \cos^3 U}$

Sol:-

Given $U = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin U = \frac{x(1 + \frac{y}{x})}{\sqrt{x}(1 + \sqrt{\frac{y}{x}})}$$

$$\sin U = x \cdot x^{-1/2} \left[\frac{1 + y/x}{1 + \sqrt{y/x}} \right]$$

$$\sin U = x^{1/2} f\left(\frac{y}{x}\right)$$

$\therefore \sin U$ is homogeneous of degree " $\frac{1}{2}$ ".

By Euler's theorem, $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \frac{1}{2} U$

$$x \frac{d}{dx}(\sin U) + y \frac{d}{dy}(\sin U) = \frac{1}{2} \sin U \rightarrow (1)$$

$$x \cdot \cos U \frac{\partial U}{\partial x} + y \cos U \frac{\partial U}{\partial y} = \frac{1}{2} \sin U$$

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \frac{1}{2} \tan U \rightarrow (2)$$

diff. w.r. to " x " partially.

$$(1) \frac{\partial U}{\partial x} + x \cdot \frac{\partial^2 U}{\partial x^2} + y \frac{\partial^2 U}{\partial x \partial y} = \frac{1}{2} \sec^2 U \cdot \frac{\partial U}{\partial x}$$

$$x \cdot \frac{\partial U}{\partial x} + x^2 \frac{\partial^2 U}{\partial x^2} + xy \frac{\partial^2 U}{\partial x \partial y} = \frac{1}{2} x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} \rightarrow (3)$$

from (2)

$$(1) y \frac{\partial U}{\partial y} + y^2 \frac{\partial^2 U}{\partial y^2} + xy \frac{\partial^2 U}{\partial x \partial y} = \frac{1}{2} \sec^2 U \cdot y \frac{\partial U}{\partial y} \rightarrow (4)$$

(3) + (4)

$$\rightarrow x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \frac{1}{2} \sec^2 U [x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y}]$$

$$x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = \left[\frac{1}{2} \sec^2 U - 1 \right] \left(x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} \right)$$

$$x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = \left(\frac{1}{2} \sec^2 U - 1 \right) \frac{1}{2} \tan U$$

$$\begin{aligned}
 &= \frac{1}{4} \frac{1}{\cos^2 u} \cdot \frac{\sin u}{\cos u} - \frac{1}{2} \tan u \\
 &= \frac{1}{4} \frac{\sin u}{\cos^3 u} - \frac{1}{2} \frac{\sin u}{\cos u} \\
 &= \frac{\sin u - 2 \sin u \cos^2 u}{4 \cos^3 u} \\
 &= \frac{\sin u (1 - 2 \cos^2 u)}{4 \cos^3 u} \\
 &= \frac{-\sin u (2 \cos^2 u - 1)}{4 \cos^3 u}
 \end{aligned}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cos 2u}{4 \cos^3 u}$$

(12) If $f(x, y) = \sqrt{x^2 - y^2} \sin^{-1}\left(\frac{y}{x}\right)$, prove that $x \frac{df}{dx} + y \frac{df}{dy} = f(x, y)$.

$$\text{Given } f(x, y) = \sqrt{x^2 - y^2} \sin^{-1}\left(\frac{y}{x}\right)$$

$$f(x, y) = x \sqrt{1 - \left(\frac{y}{x}\right)^2} \sin^{-1}\left(\frac{y}{x}\right)$$

$$f(x, y) = x \cdot f\left(\frac{y}{x}\right)$$

$\therefore f$ is homogeneous of degree "1".

By using Euler's theorem, $x \frac{du}{dx} + y \frac{du}{dy} = n \cdot u$

$$x \cdot \frac{df}{dx} + y \frac{df}{dy} = (1) f(x, y)$$

$$x \cdot \frac{df}{dx} + y \frac{df}{dy} = f(x, y)$$

(13) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$. Show that $x \frac{du}{dx} + y \frac{du}{dy} + \frac{1}{2} \cot u = 0$.

$$\text{Given } u = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\cos u = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})}$$

$$\cos u = x \cdot x^{-1/2} \left(\frac{1+y/x}{1+\sqrt{y/x}} \right)$$

$$\cos u = x^{1/2} \cdot f\left(\frac{y}{x}\right)$$

$\therefore \cos u$ is homogeneous of degree " $\frac{1}{2}$ ".

By using Euler's theorem, $x \frac{du}{dx} + y \frac{du}{dy} = n \cdot u$

$$x \cdot \frac{d}{dx} (\cos u) + y \frac{d}{dy} (\cos u) = \frac{1}{2} \cos u \rightarrow \textcircled{1}$$

$$x(-\sin u) \frac{du}{dx} + y(-\sin u) \frac{du}{dy} = \frac{1}{2} \cos u$$

$$x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = -\frac{1}{2} \cot u$$

$$\boxed{x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} + \frac{1}{2} \cot u = 0}$$

(14) If $u = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$ show that $\frac{du}{dx} = -\frac{y}{x} \cdot \frac{du}{dy}$

Sol:

Given $u = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$

$$\sin u = \frac{\sqrt{x} (1 - \sqrt{y/x})}{\sqrt{x} (1 + \sqrt{y/x})}$$

$$\sin u = x^0 \left[\frac{1 - \sqrt{y/x}}{1 + \sqrt{y/x}} \right]$$

$$\sin u = x^0 \cdot f(y/x)$$

$\therefore \sin u$ is homogeneous of degree "0".

By Euler's theorem, $x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = n \cdot u$

$$x \cdot \frac{d}{dx} (\sin u) + y \cdot \frac{d}{dy} (\sin u) = 0$$

$$x \cdot \cos u \frac{du}{dx} + y \cdot \cos u \frac{du}{dy} = 0$$

$$x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = 0$$

$$x \frac{du}{dx} = -y \frac{du}{dy}$$

$$\boxed{\frac{du}{dx} = -\frac{y}{x} \frac{du}{dy}}$$

(15) Show that $x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = 2 \log u$ where $\log u = \frac{x^3 + y^3}{3x + 4y}$

Sol:

Given $\log u = \frac{x^3 + y^3}{3x + 4y}$

$$\log u = \frac{x^3 (1 + y^3/x^3)}{x(3 + 4(y/x))}$$

$$\log u = x^2 \left[\frac{1 + (y/x)^3}{3 + 4(y/x)} \right]$$

$$\log u = x^2 \cdot f(y/x)$$

$\therefore \log u$ is homogeneous of degree "2".

By Euler's theorem, $x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = n \cdot u$

$$x \frac{d}{dx} (\log u) + y \frac{d}{dy} (\log u) = 2 \cdot \log u$$

$$x \cdot \frac{1}{u} \cdot \frac{du}{dx} + y \cdot \frac{1}{u} \cdot \frac{du}{dy} = 2 \log u$$

$$\boxed{x \frac{du}{dx} + y \frac{du}{dy} = 2u \log u.}$$

(18) If $u = (x^2 + y^2)^{1/3}$. Show that $x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = -\frac{2u}{9}$.

Sol:- Given $u = (x^2 + y^2)^{1/3}$

$$u = [x^2 (1 + y^2/x^2)]^{1/3}$$

$$u = x^{2/3} [1 + (y/x)^2]^{1/3}$$

$$u = x^{2/3} \cdot f(y/x)$$

$\therefore u$ is homogeneous of degree " $2/3$ ".

By Euler's theorem, $x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = n \cdot u$

$$x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = \frac{2}{3} u \rightarrow (1)$$

diff. w. r. to x partially

$$(1) \frac{du}{dx} + x \cdot \frac{d^2u}{dx^2} + y \cdot \frac{d^2u}{dx dy} = \frac{2}{3} \frac{du}{dx}$$

$$x \frac{du}{dx} + x^2 \frac{d^2u}{dx^2} + xy \frac{d^2u}{dx dy} = \frac{2}{3} x \frac{du}{dx} \rightarrow (2)$$

dy $y \cdot \frac{du}{dy} + y^2 \frac{d^2u}{dy^2} + xy \frac{d^2u}{dx dy} = \frac{2}{3} y \cdot \frac{du}{dy} \rightarrow (3)$

(2) + (3)

$$\Rightarrow x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} + x \frac{du}{dx} + y \frac{du}{dy} = \frac{2}{3} (x \frac{du}{dx} + y \frac{du}{dy})$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = \left(\frac{2}{3} - 1\right) (x \frac{du}{dx} + y \frac{du}{dy})$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = \left(\frac{2-3}{3}\right) \frac{2}{3} u$$

$$x^2 \frac{d^2u}{dx^2} + y^2 \frac{d^2u}{dy^2} + 2xy \frac{d^2u}{dx dy} = -\frac{2u}{9}$$

(19)

Given $u = x^2 \cdot \tan^{-1} \left(\frac{y}{x}\right) - y^2 \tan^{-1} \left(\frac{x}{y}\right)$

$$u = x^2 \tan^{-1} (y/x) - y^2 \cot^{-1} (y/x)$$

$$u = x^2 \left[\tan^{-1} (y/x) - \left(\frac{y}{x}\right)^2 \cot^{-1} (y/x) \right]$$

$$u = x^2 \cdot f(y/x)$$

$\therefore u$ is homogeneous of degree " 2 ".

By Euler's theorem, $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n \cdot U$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2U \rightarrow (1)$$

diff. w. r. to 'x' partially

$$(1) \frac{\partial U}{\partial x} + x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial x \partial y} = 2 \cdot \frac{\partial U}{\partial x}$$

$$x \cdot \frac{\partial U}{\partial x} + x^2 \cdot \frac{\partial^2 U}{\partial x^2} + xy \cdot \frac{\partial^2 U}{\partial x \partial y} = 2x \cdot \frac{\partial U}{\partial x} \rightarrow (2)$$

$$\text{dy, } y \cdot \frac{\partial U}{\partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} + xy \cdot \frac{\partial^2 U}{\partial x \partial y} = 2y \cdot \frac{\partial U}{\partial y} \rightarrow (3)$$

(2) + (3)

$$\Rightarrow x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = 2 \left(x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} \right)$$

$$x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y}$$

$$x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = 2U$$

(20)

Given $U = \operatorname{cosec}^{-1} \left(\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}} \right)^{1/2}$

$$\operatorname{cosec} U = \frac{\left(x^{1/2} (1 + y^{1/2}/x^{1/2}) \right)^{1/2}}{\left(x^{1/3} (1 + y^{1/3}/x^{1/3}) \right)}$$

$$\operatorname{cosec} U = \frac{x^{1/4}}{x^{1/6}} \cdot \left(\frac{1 + (y/x)^{1/2}}{1 + (y/x)^{1/3}} \right)^{1/2}$$

$$\operatorname{cosec} U = x^{1/4} \cdot x^{-1/6} \cdot f(y/x)$$

$$\operatorname{cosec} U = x^{1/12} \cdot f(y/x)$$

$$\frac{2(6-4)}{3 \cdot 2}$$

$$\frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

$\therefore \operatorname{cosec} U$ is homogeneous of degree ' $1/12$ '.

By Euler's theorem, $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n \cdot U$

$$x \cdot \frac{\partial}{\partial x} (\operatorname{cosec} U) + y \cdot \frac{\partial}{\partial y} (\operatorname{cosec} U) = \frac{1}{12} \cdot \operatorname{cosec} U$$

$$-x \cdot \operatorname{cosec} U \cdot \cot U \cdot \frac{\partial U}{\partial x} + y (-\operatorname{cosec} U \cdot \cot U) \frac{\partial U}{\partial y} = \frac{1}{12} \operatorname{cosec} U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{1}{12} \frac{\operatorname{cosec} U}{-\operatorname{cosec} U \cdot \cot U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = -\frac{1}{12} \tan U \rightarrow (1)$$

diff. w. r. to "x" partially,

$$(1) \frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{12} \sec^2 u \cdot \frac{\partial u}{\partial x}$$

$$x \cdot \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{12} \sec^2 u \cdot x \cdot \frac{\partial u}{\partial x} \rightarrow (2)$$

$$(2) \quad y \cdot \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{12} \sec^2 u \cdot y \cdot \frac{\partial u}{\partial y} \rightarrow (3)$$

$$(2) + (3) \Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{12} \sec^2 u (x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y})$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{1}{12} \sec^2 u - 1 \right) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$= \left(\frac{1}{12} \sec^2 u - 1 \right) \left(\frac{1}{12} \tan u \right)$$

$$= \frac{1}{144} \cdot \frac{\sin u}{\cos^3 u} + \frac{1}{12} \cdot \frac{\sin u}{\cos u}$$

$$= \frac{\sin u + 12 \sin u \cos^2 u}{144 \cos^3 u}$$

$$= \frac{\sin u (1 + 12 \cos^2 u)}{144 \cos^3 u}$$

$$= \frac{\sin u (1 + 12 (1 - \sin^2 u))}{144 \cos^3 u}$$

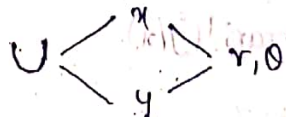
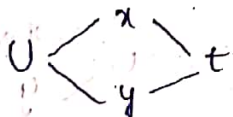
$$= \frac{\sin u (1 + 12 - 12 \sin^2 u)}{144 \cos^3 u}$$

$$= \frac{11 \sin u - 12 \sin^3 u}{144 \cos^3 u}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{11}{144} \frac{\sin u}{\cos^3 u} - \frac{1}{12} \tan^3 u$$

22/11/19
Friday

Total Derivative and Chain Rule:

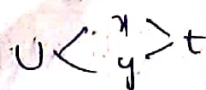


① If $u = \sin^{-1}(x-y)$, $x=3t$, $y=4t^3$. show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Sol:-

Given $u = \sin^{-1}(x-y)$, $x=3t$, $y=4t^3$

By using Total derivative



$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{d}{dx} \sin^{-1}(x-y) = \frac{1}{\sqrt{1-(x-y)^2}} (1-0) = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{d}{dy} [\sin^{-1}(x-y)] = \frac{1}{\sqrt{1-(x-y)^2}} (0-1) = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{dx}{dt} = \frac{d}{dt}(3t) = 3, \quad \frac{dy}{dt} = \frac{d}{dt}(4t^3) = 12t^2$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}} (3) + \frac{-1}{\sqrt{1-(x-y)^2}} (12t^2)$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-x^2-y^2+2xy}}$$

$$= \frac{3-12t^2}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$= \frac{3(1-4t^2)}{\sqrt{-16t^6+24t^4-9t^2+1}}$$

$$= \frac{3(1-4t^2)}{\sqrt{-16x^3+24x^2-9x+1}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-x)(1-4x)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-x^2}(1-4x)}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-t^2}(1-4t^2)}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

$$\begin{array}{r|rrrr} 1 & -16 & 24 & -9 & 1 \\ & 0 & -16 & 8 & -1 \\ & -16 & 8 & -1 & 0 \end{array}$$

$$(x-1)(16x^2+8x-1)=0$$

$$(x-1)[(16x^2-8x+1)]=0$$

$$(1-x)(4x-1)^2=0$$

⑩ If $u = \tan^{-1}(y/x)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ then find $\frac{du}{dt}$.

Sol: Given $u = \tan^{-1}(y/x)$

By using Total Derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial}{\partial x} [\tan^{-1}(y/x)] = \frac{1}{1 + \frac{y^2}{x^2}} \cdot y \left(\frac{-1}{x^2} \right) = \frac{-y}{x^2} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial}{\partial y} [\tan^{-1}(y/x)] = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{1}{x} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} = \frac{x}{x^2+y^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t - e^{-t}) = e^t - e^{-t}(-1) = e^t + e^{-t}$$

$$u = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t + e^{-t}) = e^t + e^{-t}(-1) = e^t - e^{-t}$$

$$\frac{du}{dt} = \frac{-y}{x^2+y^2}(e^t + e^{-t}) + \frac{x}{x^2+y^2}(e^t - e^{-t})$$

$$= \frac{-y(y) + x(x)}{x^2+y^2} = \frac{x^2 - y^2}{x^2+y^2}$$

$$= \frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$$

$$= \frac{e^{2t} + e^{-2t} - 2 - e^{2t} - e^{-2t} - 2}{e^{2t} + e^{-2t} - 2 + e^{2t} + e^{-2t} + 2}$$

$$= \frac{-4}{2(e^{2t} + e^{-2t})}$$

$$= -\frac{1}{e^{2t} + e^{-2t}} = -\frac{1}{2 \cosh 2t} = -\operatorname{sech} 2t$$

Q4 If $u = f(x^2 + 2yz, y^2 + 2zx)$ prove that $(y^2 - zx) \frac{du}{dx} + (x^2 - yz) \frac{du}{dy} + (z^2 - xy) \frac{du}{dz} = 0$.

Solr Given $u = f(x^2 + 2yz, y^2 + 2zx)$

$u = f(r, s)$ where $r = x^2 + 2yz, s = y^2 + 2zx$

By using chain rule,

$$\frac{du}{dx} = \frac{du}{dr} \cdot \frac{dr}{dx} + \frac{du}{ds} \cdot \frac{ds}{dx}$$

$$\frac{du}{dy} = \frac{du}{dr} \cdot \frac{dr}{dy} + \frac{du}{ds} \cdot \frac{ds}{dy}$$

$$\frac{du}{dz} = \frac{du}{dr} \cdot \frac{dr}{dz} + \frac{du}{ds} \cdot \frac{ds}{dz}$$

$$\left[\frac{du}{dr} = \frac{df}{dr}; \frac{du}{ds} = \frac{df}{ds} \right]$$

$$u = f(r, s)$$

$$\Rightarrow \frac{dr}{dx} = \frac{d}{dx}(x^2 + 2yz) = 2x + 0 = 2x$$

$$\Rightarrow \frac{dr}{dy} = \frac{d}{dy}(x^2 + 2yz) = 0 + 2z = 2z$$

$$\Rightarrow \frac{dr}{dz} = \frac{d}{dz}(x^2 + 2yz) = 0 + 2y = 2y$$

$$\Rightarrow \frac{ds}{dx} = \frac{d}{dx}(y^2 + 2zx) = 0 + 2z = 2z$$

$$\Rightarrow \frac{ds}{dy} = \frac{d}{dy}(y^2 + 2zx) = 2y + 0 = 2y$$

$$\Rightarrow \frac{ds}{dz} = \frac{d}{dz}(y^2 + 2zx) = 0 + 2x = 2x$$

$$\frac{dU}{dx} = \frac{df}{dx}(2x) + \frac{df}{dz}(2z)$$

$$\frac{dU}{dy} = \frac{df}{dy}(2z) + \frac{df}{dz}(2y)$$

$$\frac{dU}{dz} = \frac{df}{dz}(2y) + \frac{df}{dz}(2x)$$

Now, $(y^2 - zx) \frac{dU}{dx} + (x^2 - yz) \frac{dU}{dy} + (z^2 - xy) \frac{dU}{dz}$

$$= (y^2 - zx) \left(\frac{df}{dx} 2x + \frac{df}{dz} 2z \right) + (x^2 - yz) \left(\frac{df}{dy} 2z + \frac{df}{dz} 2y \right) + (z^2 - xy) \left(\frac{df}{dz} 2y + \frac{df}{dz} 2x \right)$$

$$= 2xy^2 \frac{df}{dx} - 2xz^2 \frac{df}{dx} + 2zy^2 \frac{df}{dz} - 2z^2x \frac{df}{dz} + 2xz^2 \frac{df}{dy} - 2z^2y \frac{df}{dy} + 2yx^2 \frac{df}{dz} - 2yz^2 \frac{df}{dz} + 2yz^2 \frac{df}{dz} - 2xy^2 \frac{df}{dz} + 2xz^2 \frac{df}{dz} - 2xy^2 \frac{df}{dz}$$

$$= 0.$$

② If z is a function of x and y where $x = e^u + e^{-v}$ and $y = e^v - e^u$. show that $\frac{dz}{du} - \frac{dz}{dv} = x \frac{dz}{dx} - y \frac{dz}{dy}$.

Sol:

Given $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^v - e^u$.

By using chain Rule,

$$z = f(x, y)_{u, v}$$

$$\frac{dz}{du} = \frac{dz}{dx} \cdot \frac{dx}{du} + \frac{dz}{dy} \cdot \frac{dy}{du}$$

$$\frac{dz}{dv} = \frac{dz}{dx} \cdot \frac{dx}{dv} + \frac{dz}{dy} \cdot \frac{dy}{dv}$$

$$\frac{dz}{dx} = \frac{df}{dx}, \quad \frac{dz}{dy} = \frac{df}{dy}$$

$$\frac{dz}{dx}$$

$$\frac{dx}{du} = e^u + 0 = e^u$$

$$\frac{dx}{dv} = 0 - e^{-v} = -e^{-v}$$

$$\frac{dy}{du} = e^{-v} - 0 = e^{-v}$$

$$\frac{dy}{dv} = 0 - e^u = -e^u$$

$$\frac{dz}{du} = \frac{df}{dx}(e^u) + \frac{df}{dy}(e^{-v}) = \frac{df}{dx} e^u - \frac{df}{dy} e^{-v}$$

$$\frac{dz}{dv} = \frac{df}{dx}(-e^{-v}) + \frac{df}{dy}(-e^u) = -\left(\frac{df}{dx} e^{-v} + \frac{df}{dy} e^u \right)$$

$$\begin{aligned}
 \therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} &= \frac{\partial f}{\partial x} e^u - \frac{\partial f}{\partial y} e^{-v} + \frac{\partial f}{\partial x} e^v + \frac{\partial f}{\partial y} e^v \\
 &= (e^u + e^{-v}) \frac{\partial f}{\partial x} + (e^v - e^{-u}) \frac{\partial f}{\partial y} \\
 &= (e^u + e^{-v}) \frac{\partial f}{\partial x} - (e^{-u} - e^v) \frac{\partial f}{\partial y} \\
 &= x \cdot \frac{\partial z}{\partial x} - y \cdot \frac{\partial z}{\partial y}.
 \end{aligned}$$

③ If $U = f(y-z, z-x, x-y)$ prove that $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$

Given $U = f(y-z, z-x, x-y)$

$$U = f(a, b, c)$$

Where $a = y-z$, $b = z-x$, $c = x-y$

By using chain Rule, $U \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow x, y, z$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial U}{\partial b} \cdot \frac{\partial b}{\partial x} + \frac{\partial U}{\partial c} \cdot \frac{\partial c}{\partial x}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial U}{\partial b} \cdot \frac{\partial b}{\partial y} + \frac{\partial U}{\partial c} \cdot \frac{\partial c}{\partial y}$$

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial a} \cdot \frac{\partial a}{\partial z} + \frac{\partial U}{\partial b} \cdot \frac{\partial b}{\partial z} + \frac{\partial U}{\partial c} \cdot \frac{\partial c}{\partial z}$$

$$\frac{\partial U}{\partial a} = \frac{\partial f}{\partial a}, \quad \frac{\partial U}{\partial b} = \frac{\partial f}{\partial b}, \quad \frac{\partial U}{\partial c} = \frac{\partial f}{\partial c}$$

$$\begin{array}{l|l|l}
 \frac{\partial a}{\partial x} = \frac{\partial}{\partial x}(y-z) = 0 & \frac{\partial b}{\partial x} = \frac{\partial}{\partial x}(z-x) = -1 & \frac{\partial c}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1 \\
 \frac{\partial a}{\partial y} = \frac{\partial}{\partial y}(y-z) = 1 & \frac{\partial b}{\partial y} = \frac{\partial}{\partial y}(z-x) = 0 & \frac{\partial c}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1 \\
 \frac{\partial a}{\partial z} = \frac{\partial}{\partial z}(y-z) = -1 & \frac{\partial b}{\partial z} = \frac{\partial}{\partial z}(z-x) = 1 & \frac{\partial c}{\partial z} = \frac{\partial}{\partial z}(x-y) = 0
 \end{array}$$

$$\frac{\partial U}{\partial x} = \frac{\partial f}{\partial a}(0) + \frac{\partial f}{\partial b}(-1) + \frac{\partial f}{\partial c}(1) = -\frac{\partial f}{\partial b} + \frac{\partial f}{\partial c}$$

$$\frac{\partial U}{\partial y} = \frac{\partial f}{\partial a}(1) + \frac{\partial f}{\partial b}(0) + \frac{\partial f}{\partial c}(-1) = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial c}$$

$$\frac{\partial U}{\partial z} = \frac{\partial f}{\partial a}(-1) + \frac{\partial f}{\partial b}(1) + \frac{\partial f}{\partial c}(0) = -\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}$$

$$\therefore \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$$

$$= -\frac{\partial f}{\partial b} + \frac{\partial f}{\partial c} + \frac{\partial f}{\partial a} - \frac{\partial f}{\partial c} - \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}$$

$$= 0$$

④ If $w = f(x, y)$, $x = r \cos \theta$, $y = r \sin \theta$. Show that

$$\left(\frac{dw}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{dw}{d\theta}\right)^2 = \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2$$

Soln

Given $w = f(x, y)$

and $x = r \cos \theta$, $y = r \sin \theta$.

By using chain Rule,

$$w \left\langle \begin{matrix} x \\ y \end{matrix} \right\rangle r, \theta$$

$$\frac{dw}{dr} = \frac{dw}{dx} \cdot \frac{dx}{dr} + \frac{dw}{dy} \cdot \frac{dy}{dr}$$

$$\frac{dw}{d\theta} = \frac{dw}{dx} \cdot \frac{dx}{d\theta} + \frac{dw}{dy} \cdot \frac{dy}{d\theta}$$

$$\frac{dw}{dx} = \frac{df}{dx}, \quad \frac{dw}{dy} = \frac{df}{dy}$$

$$\frac{dx}{dr} = \frac{d}{dr}(r \cos \theta) = \cos \theta \quad \left| \quad \frac{dy}{dr} = \frac{d}{dr}(r \sin \theta) = \sin \theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r \cos \theta) = -r \sin \theta \quad \left| \quad \frac{dy}{d\theta} = \frac{d}{d\theta}(r \sin \theta) = r \cos \theta$$

$$\frac{dw}{dr} = \frac{df}{dx} (\cos \theta) + \frac{df}{dy} \sin \theta \rightarrow \textcircled{1}$$

$$\frac{dw}{d\theta} = \frac{df}{dx} (-r \sin \theta) + \frac{df}{dy} r \cos \theta \rightarrow \textcircled{2}$$

$$\textcircled{1} \Rightarrow \left(\frac{dw}{dr}\right)^2 = \left(\frac{df}{dx}\right)^2 \cos^2 \theta + \left(\frac{df}{dy}\right)^2 \sin^2 \theta + 2 \frac{df}{dx} \frac{df}{dy} \sin \theta \cos \theta$$

$$\textcircled{2} \Rightarrow \left(\frac{dw}{d\theta}\right)^2 = \left(\frac{df}{dx}\right)^2 r^2 \sin^2 \theta + \left(\frac{df}{dy}\right)^2 r^2 \cos^2 \theta - 2 r^2 \frac{df}{dx} \frac{df}{dy} \sin \theta \cos \theta$$

$$\left(\frac{dw}{d\theta}\right)^2 = r^2 \left[\left(\frac{df}{dx}\right)^2 \sin^2 \theta + \left(\frac{df}{dy}\right)^2 \cos^2 \theta - 2 \frac{df}{dx} \frac{df}{dy} \sin \theta \cos \theta \right]$$

$$\frac{1}{r^2} \left(\frac{dw}{d\theta}\right)^2 = \left(\frac{df}{dx}\right)^2 \sin^2 \theta + \left(\frac{df}{dy}\right)^2 \cos^2 \theta - 2 \frac{df}{dx} \frac{df}{dy} \sin \theta \cos \theta$$

$$\begin{aligned} \textcircled{1} + \textcircled{3} \Rightarrow \left(\frac{dw}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{dw}{d\theta}\right)^2 &= \left(\frac{df}{dx}\right)^2 [\cos^2 \theta + \sin^2 \theta] + \left(\frac{df}{dy}\right)^2 [\sin^2 \theta + \cos^2 \theta] \\ &= \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 \end{aligned} \rightarrow \textcircled{3}$$

⑤ If f is the function u, v and $u = x^2 - y^2$, $v = 2xy$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2) \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right)$

Soln

Given $\phi = f(u, v)$ $f = \phi(u, v)$

$u = x^2 - y^2$, $v = 2xy$

By using chain Rule,

$$\phi \left\langle \begin{matrix} u \\ v \end{matrix} \right\rangle x, y$$

$$\frac{df}{dx} = \frac{df}{dU} \cdot \frac{dU}{dx} + \frac{df}{dV} \cdot \frac{dV}{dx}$$

$$\frac{df}{dy} = \frac{df}{dU} \cdot \frac{dU}{dy} + \frac{df}{dV} \cdot \frac{dV}{dy}$$

$$\frac{df}{dU} = \frac{d\phi}{dU}, \quad \frac{df}{dV} = \frac{d\phi}{dV}$$

$$\begin{aligned} \frac{dU}{dx} &= \frac{d}{dx} (x^2 - y^2) = 2x & \left| \quad \frac{dV}{dx} &= \frac{d}{dx} (2xy) = 2y \\ \frac{dU}{dy} &= \frac{d}{dy} (x^2 - y^2) = -2y & \left| \quad \frac{dV}{dy} &= \frac{d}{dy} (2xy) = 2x \end{aligned}$$

$$\frac{df}{dx} = \frac{d\phi}{dU} (2x) + \frac{d\phi}{dV} (2y)$$

$$\frac{df}{dy} = \frac{d\phi}{dU} (-2y) + \frac{d\phi}{dV} (2x)$$

$$\frac{df}{dx} = 2 \frac{d\phi}{dU} x + 2 \frac{d\phi}{dV} y$$

diff. w. r. to "x" partially

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= 2 \frac{d}{dx} \left(\frac{d\phi}{dU} \right) + 2x \frac{d^2\phi}{dU dx} + 2y \frac{d^2\phi}{dV dx} + 2 \frac{d}{dx} \left(\frac{d\phi}{dV} \right) \\ &= 2 \frac{d}{dx} \left(\frac{d\phi}{dU} \right) + 2x \frac{d^2\phi}{dU dx} + 2y \frac{d^2\phi}{dV dx} \end{aligned}$$

$$\frac{d^2\phi}{dy^2} = 2 \frac{d}{dy} \left(\frac{d\phi}{dU} \right) + 2x \frac{d^2\phi}{dU dy} + 2y \frac{d^2\phi}{dV dy}$$

$$\frac{df}{dx} = 2x \cdot \frac{d\phi}{dU} + 2y \cdot \frac{d\phi}{dV} \rightarrow \textcircled{1}$$

$$\frac{df}{dx} = 2 \left[x \cdot \frac{d\phi}{dU} + y \cdot \frac{d\phi}{dV} \right]$$

$$\frac{df}{dx} = 2 \left[x \cdot \frac{d}{dx} \left(\frac{d\phi}{dU} \right) + y \cdot \frac{d}{dx} \left(\frac{d\phi}{dV} \right) \right]$$

$$\frac{d}{dx} = 2 \left[x \cdot \frac{d}{dx} \left(\frac{d\phi}{dU} \right) + y \cdot \frac{d}{dx} \left(\frac{d\phi}{dV} \right) \right] \rightarrow \textcircled{2}$$

$$\Rightarrow \frac{d^2f}{dx^2} = \frac{d}{dx} \left(\frac{df}{dx} \right)$$

$$= 2 \left[x \cdot \frac{d}{dx} \left(\frac{d\phi}{dU} \right) + y \cdot \frac{d}{dx} \left(\frac{d\phi}{dV} \right) \right] \left[\because \text{from } \textcircled{1} \text{ \& } \textcircled{2} \right]$$

$$= 4 \left(x^2 \cdot \frac{d^2\phi}{dU^2} + xy \cdot \frac{d^2\phi}{dU dV} + xy \cdot \frac{d^2\phi}{dV dU} + y^2 \cdot \frac{d^2\phi}{dV^2} \right)$$

$$\frac{d^2f}{dx^2} = 4 \left(x^2 \cdot \frac{d^2\phi}{dU^2} + 2xy \cdot \frac{d^2\phi}{dU dV} + y^2 \cdot \frac{d^2\phi}{dV^2} \right) \rightarrow \textcircled{3}$$

⑥ If $U = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t} \cos 3t$, $z = e^{2t} \sin 3t$ find $\frac{dU}{dt}$.

Solr

Given $U = x^2 + y^2 + z^2$

and $x = e^{2t}$, $y = e^{2t} \cos 3t$, $z = e^{2t} \sin 3t$

By using Total Derivative,

$$U \begin{matrix} \swarrow x \\ \searrow y \\ \searrow z \end{matrix} t$$

$$\frac{dU}{dt} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2 + z^2) = 2x$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y} (x^2 + y^2 + z^2) = 2y$$

$$\frac{\partial U}{\partial z} = \frac{\partial}{\partial z} (x^2 + y^2 + z^2) = 2z$$

$$\frac{dx}{dt} = \frac{d}{dt} (e^{2t}) = 2e^{2t}$$

$$\frac{dy}{dt} = \frac{d}{dt} (e^{2t} \cos 3t) = e^{2t} (3 \cos 3t - \sin 3t)$$

$$\frac{dz}{dt} = \frac{d}{dt} (e^{2t} \sin 3t)$$

$$= e^{2t} \cos 3t (3) + \sin 3t (e^{2t}) = 3e^{2t} \cos 3t + e^{2t} \sin 3t$$

$$= -3e^{2t} \sin 3t + 2e^{2t} \cos 3t$$

$$\frac{dU}{dt} = 2x(2e^{2t}) + 2y(-3e^{2t} \sin 3t + 2e^{2t} \cos 3t) + 2z(3e^{2t} \cos 3t + e^{2t} \sin 3t)$$

$$= 4x e^{2t} - 6y e^{2t} \sin 3t + 4y e^{2t} \cos 3t + 6z e^{2t} \cos 3t + 4z e^{2t} \sin 3t$$

$$= 4x e^{2t} - e^{2t} \sin 3t (6y - 4z) + e^{2t} \cos 3t (4y + 6z)$$

$$= 4x e^{2t} - e^{2t} \sin 3t (6e^{2t} \cos 3t - 4e^{2t} \sin 3t) + e^{2t} \cos 3t (4e^{2t} \cos 3t + 6e^{2t} \sin 3t)$$

$$= 4x e^{2t} - 6e^{4t} \sin 3t \cos 3t + 4e^{4t} \sin^2 3t + 4e^{4t} \cos^2 3t + 6e^{4t} \sin 3t \cos 3t$$

$$= 4e^{4t} (1 + \sin^2 3t + \cos^2 3t)$$

$$= 4e^{4t} (1 + 1) = 4e^{4t} (2) = 8e^{4t}$$

⑦ If $U = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ then find $\frac{dU}{dt}$.

Given $U = \sin\left(\frac{x}{y}\right)$

$x = e^t$, $y = t^2$

By using Total Derivative,

$$U \begin{matrix} \swarrow x \\ \searrow y \end{matrix} t$$

$$\frac{dU}{dt} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial U}{\partial x} = \cos\left(\frac{x}{y}\right) \left(\frac{1}{y}\right)$$

$$\frac{\partial U}{\partial y} = \cos\left(\frac{x}{y}\right) x \left(-\frac{1}{y^2}\right)$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dU}{dt} = \frac{1}{y} \cos\left(\frac{x}{y}\right) e^t - \frac{x}{y^2} \cos\left(\frac{x}{y}\right) 2t$$

$$= \frac{e^t}{t^2} \cos\left(\frac{et}{t^2}\right) - \frac{et}{t^4} \cos\left(\frac{et}{t^2}\right) 2t$$

$$= \frac{et}{t^2} \left\{ \cos\left(\frac{et}{t^2}\right) \left[1 - \frac{2t}{t^2}\right] \right\}$$

$$= \frac{et}{t^2} \cos\left(\frac{et}{t^2}\right) \left(\frac{t^2 - 2t}{t^2}\right)$$

$$\frac{du}{dt} = \frac{et(t^2 - 2t)}{t^4} \cos\left(\frac{et}{t^2}\right) \Rightarrow \frac{du}{dt} = \frac{e^t(t-2)}{t^3} \cdot \cos\left(\frac{et}{t^2}\right)$$

⑧ If $u = x^3 + y^3$ where $x = a \cos t$, $y = b \sin t$ find $\frac{du}{dt}$.

Given $u = x^3 + y^3$, $x = a \cos t$, $y = b \sin t$.

By using Total Derivative, $u \leftarrow x, y \rightarrow t$

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$$

$$\frac{du}{dx} = 3x^2$$

$$\frac{du}{dy} = 3y^2$$

$$\frac{dx}{dt} = -a \sin t$$

$$\frac{dy}{dt} = b \cos t$$

$$\frac{du}{dt} = 3x^2 (-a \sin t) + 3y^2 \cdot b \cos t$$

$$= -3(x^2 a \sin t + y^2 b \cos t)$$

$$= -3(a^2 \cos^2 t \cdot a \sin t + b^2 \sin^2 t \cdot b \cos t)$$

$$= -3(a^3 \sin t \cdot \cos^2 t + b^3 \sin^2 t \cdot \cos t)$$

$$= -3 \sin t \cdot \cos t (a^3 \cos t + b^3 \sin t)$$

$$= -\frac{3}{2} \sin 2t (a^3 \cos t + b^3 \sin t)$$

⑨ If $z = u^2 + v^2$, $u = r \cos \theta$, $v = r \sin \theta$ find $\frac{dz}{dr}$, $\frac{dz}{d\theta}$.

Sol Given $z = u^2 + v^2 = f(u, v)$

$$u = r \cos \theta, \quad v = r \sin \theta$$

By using chain Rule, $\begin{matrix} u \\ v \end{matrix} \leftarrow r, \theta$

$$\frac{dz}{dr} = \frac{dz}{du} \cdot \frac{du}{dr} + \frac{dz}{dv} \cdot \frac{dv}{dr}$$

$$\frac{dz}{d\theta} = \frac{dz}{du} \cdot \frac{du}{d\theta} + \frac{dz}{dv} \cdot \frac{dv}{d\theta}$$

$$\frac{dz}{du} = 2u$$

$$\frac{dz}{dv} = 2v$$

$$\frac{du}{dr} = \cos \theta$$

$$\frac{du}{d\theta} = r(-\sin \theta)$$

$$\frac{dv}{dr} = \sin \theta$$

$$\frac{dv}{d\theta} = r \cos \theta$$

$$\begin{aligned}\frac{dz}{dr} &= 2u \cos \theta + 2v (-r \sin \theta) \\ &= 2(r \cos \theta) \cos \theta + -2(r \sin \theta)(-r \sin \theta) \\ &= 2r \cos^2 \theta + 2r^2 \sin^2 \theta \\ &= 2r (\cos^2 \theta + r \sin^2 \theta)\end{aligned}$$

$$\begin{aligned}\frac{dz}{d\theta} &= 2u \sin \theta + 2v (r \cos \theta) \\ &= 2(r \cos \theta)(\sin \theta) + 2(r \sin \theta) r \cos \theta \\ &= r \cdot 2 \sin \theta \cos \theta + r^2 \cdot 2 \sin \theta \cos \theta \\ &= r \cdot \sin 2\theta + r^2 \cdot \sin 2\theta \\ &= r \sin 2\theta (1+r)\end{aligned}$$

⑩ If $u = \tan^{-1}(y/x)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ find $\frac{du}{dt}$.

Given $u = \tan^{-1}(y/x)$

$$x = e^t - e^{-t}$$

⑪ If $z = \log(u^2 + v)$, $u = e^{x^2 + y^2}$, $v = x^2 + y$ find $\frac{dz}{dx}$, $\frac{dz}{dy}$.

Given $z = \log(u^2 + v)$

$$u = e^{x^2 + y^2}, \quad v = x^2 + y$$

By using chain Rule, $z = \log \left(\frac{u}{v} \right) \cdot x, y$

$$\frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx} + \frac{dz}{dv} \cdot \frac{dv}{dx}$$

$$\frac{dz}{dy} = \frac{dz}{du} \cdot \frac{du}{dy} + \frac{dz}{dv} \cdot \frac{dv}{dy}$$

$$\frac{dz}{du} = \frac{1}{u^2 + v} (2u)$$

$$\frac{du}{dx} = e^{x^2 + y^2} (2x)$$

$$\frac{dv}{dx} = 2x$$

$$\frac{dz}{dv} = \frac{1}{u^2 + v} (1)$$

$$\frac{du}{dy} = e^{x^2 + y^2} (2y)$$

$$\frac{dv}{dy} = 1$$

$$\frac{dz}{dx} = \frac{2u}{u^2 + v} \cdot e^{x^2 + y^2} (2x) + \frac{1}{u^2 + v} (2x)$$

$$= \frac{2x}{u^2 + v} (2u \cdot e^{x^2 + y^2} + 1)$$

$$= \frac{2x}{(e^{x^2 + y^2})^2 + x^2 + y} (2 \cdot e^{x^2 + y^2} \cdot e^{x^2 + y^2} + 1)$$

$$= \frac{2x}{e^{2(x^2 + y^2)} + x^2 + y} [2 \cdot e^{2(x^2 + y^2)} + 1]$$

$$\frac{dz}{dy} = \frac{2v}{v^2+1} e^{x^2+y^2} (2y) + \frac{1}{v^2+1} (1)$$

$$= \frac{4yv \cdot e^{x^2+y^2}}{v^2+1} + \frac{1}{v^2+1}$$

$$= \frac{4yv e^{x^2+y^2} + 1}{v^2(e^{x^2+y^2})^2 + x^2+y^2}$$

$$= \frac{4y \cdot e^{2(x^2+y^2)} + 1}{e^{2(x^2+y^2)} + x^2+y^2}$$

(12) If $U = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ prove that,

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} + z \frac{\partial U}{\partial z} = 0.$$

Given $U = f(r, s, t)$

$r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$.

By using chain Rule,

$$U = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial U}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial U}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial U}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial U}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial U}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial U}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$\frac{\partial U}{\partial r} = \frac{df}{dr}$$

$$\frac{\partial r}{\partial x} = \frac{d}{dx} \left(\frac{x}{y} \right) = \frac{1}{y}$$

$$\frac{\partial s}{\partial x} = \frac{d}{dx} \left(\frac{y}{z} \right) = 0$$

$$\frac{\partial t}{\partial x} = \frac{d}{dx} \left(\frac{z}{x} \right) = -\frac{z}{x^2}$$

$$\frac{\partial U}{\partial s} = \frac{df}{ds}$$

$$\frac{\partial r}{\partial y} = \frac{d}{dy} \left(\frac{x}{y} \right) = -\frac{x}{y^2}$$

$$\frac{\partial s}{\partial y} = \frac{d}{dy} \left(\frac{y}{z} \right) = \frac{1}{z}$$

$$\frac{\partial t}{\partial y} = \frac{d}{dy} \left(\frac{z}{x} \right) = 0$$

$$\frac{\partial U}{\partial t} = \frac{df}{dt}$$

$$\frac{\partial r}{\partial z} = \frac{d}{dz} \left(\frac{x}{y} \right) = 0$$

$$\frac{\partial s}{\partial z} = \frac{d}{dz} \left(\frac{y}{z} \right) = -\frac{y}{z^2}$$

$$\frac{\partial t}{\partial z} = \frac{d}{dz} \left(\frac{z}{x} \right) = \frac{1}{x}$$

$$\frac{\partial U}{\partial x} = \frac{df}{dr} \left(\frac{1}{y} \right) + \frac{df}{ds} (0) + \frac{df}{dt} \left(-\frac{z}{x^2} \right) = \frac{1}{y} \frac{df}{dr} - \frac{z}{x^2} \frac{df}{dt}$$

$$\Rightarrow x \frac{\partial U}{\partial x} = \frac{x}{y} \frac{df}{dr} - \frac{z}{x} \frac{df}{dt} \rightarrow \textcircled{1}$$

$$\frac{\partial U}{\partial y} = \frac{df}{dr} \left(-\frac{x}{y^2} \right) + \frac{df}{ds} \left(\frac{1}{z} \right) + \frac{df}{dt} (0)$$

$$\Rightarrow y \frac{\partial U}{\partial y} = -\frac{x}{y} \frac{df}{dr} + \frac{y}{z} \frac{df}{ds} \rightarrow \textcircled{2}$$

$$\frac{\partial U}{\partial z} = \frac{df}{dr} (0) + \frac{df}{ds} \left(-\frac{y}{z^2} \right) + \frac{df}{dt} \left(\frac{1}{x} \right)$$

$$\Rightarrow z \cdot \frac{\partial u}{\partial z} = -\frac{y}{z} \frac{\partial f}{\partial s} + \frac{z}{x} \frac{\partial f}{\partial t} \rightarrow \textcircled{3}$$

Adding ①+②+③

$$\begin{aligned} \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \\ = \frac{x}{y} \frac{\partial f}{\partial r} - \frac{z}{x} \frac{\partial f}{\partial t} + \frac{x}{y} \frac{\partial f}{\partial r} + \frac{y}{z} \frac{\partial f}{\partial s} - \frac{y}{z} \frac{\partial f}{\partial s} + \frac{z}{x} \frac{\partial f}{\partial t} \\ = 0. \end{aligned}$$

⑮ If $U = f(r, s)$, $r = x+y$, $s = x-y$. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial r}$.

Given $U = f(r, s)$

$$r = x+y, \quad s = x-y$$

By using chain Rule,

$$U \left(\begin{matrix} r \\ s \end{matrix} \right) \text{ where } r = x+y, s = x-y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial r} = \frac{df}{dr}$$

$$\frac{\partial u}{\partial s} = \frac{df}{ds}$$

$$\frac{\partial r}{\partial x} = \frac{d}{dx}(x+y) = 1$$

$$\frac{\partial r}{\partial y} = \frac{d}{dy}(x+y) = 1$$

$$\frac{\partial s}{\partial x} = \frac{d}{dx}(x-y) = 1$$

$$\frac{\partial s}{\partial y} = \frac{d}{dy}(x-y) = -1$$

$$\frac{\partial u}{\partial x} = \frac{df}{dr} (1) + \frac{df}{ds} (1) = \frac{df}{dr} + \frac{df}{ds}$$

$$\frac{\partial u}{\partial y} = \frac{df}{dr} (1) + \frac{df}{ds} (-1) = \frac{df}{dr} - \frac{df}{ds}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{df}{dr} + \frac{df}{ds} + \frac{df}{dr} - \frac{df}{ds}$$

$$= 2 \cdot \frac{df}{dr}$$

$$= 2 \cdot \frac{\partial u}{\partial r}$$

⑯ If $U = f(2x-3y, 3y-4z, 4z-2x)$ Prove that

$$\frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z} = 0.$$

$$\text{Given } U = f(2x-3y, 3y-4z, 4z-2x)$$

$$U = f(r, s, t)$$

where $r = 2x - 3y$, $s = 3y - 4z$, $t = 4z - 2x$.

By using Chain Rule,

$$u \leftarrow \begin{matrix} r \\ s \\ t \end{matrix} \rightarrow x, y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}$$

$$\frac{\partial r}{\partial x} = \frac{d}{dx}(2x - 3y) = 2$$

$$\frac{\partial r}{\partial y} = \frac{d}{dy}(2x - 3y) = -3$$

$$\frac{\partial r}{\partial z} = \frac{d}{dz}(2x - 3y) = 0$$

$$\frac{\partial s}{\partial x} = \frac{d}{dx}(3y - 4z) = 0$$

$$\frac{\partial s}{\partial y} = \frac{d}{dy}(3y - 4z) = 3$$

$$\frac{\partial s}{\partial z} = \frac{d}{dz}(3y - 4z) = -4$$

$$\frac{\partial t}{\partial x} = \frac{d}{dx}(4z - 2x) = -2$$

$$\frac{\partial t}{\partial y} = \frac{d}{dy}(4z - 2x) = 0$$

$$\frac{\partial t}{\partial z} = \frac{d}{dz}(4z - 2x) = 4$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-2)$$

$$\Rightarrow \frac{1}{2} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \rightarrow \textcircled{1}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + \frac{\partial f}{\partial t}(0)$$

$$\Rightarrow \frac{1}{3} \frac{\partial u}{\partial y} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \rightarrow \textcircled{2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-4) + \frac{\partial f}{\partial t}(4)$$

$$\Rightarrow \frac{1}{4} \frac{\partial u}{\partial z} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \rightarrow \textcircled{3}$$

Adding $\textcircled{1} + \textcircled{2} + \textcircled{3}$

$$\Rightarrow \frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z}$$

$$= \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} - \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} - \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}$$

$$= 0.$$

$$\therefore \frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z} = 0.$$

⑤ → Continuous

$$\frac{df}{dy} = -2y \frac{d\theta}{dU} + 2x \frac{d\theta}{dV} \rightarrow \textcircled{4}$$

$$\frac{df}{dy} = 2 \left(x \cdot \frac{d\theta}{dV} - y \cdot \frac{d\theta}{dU} \right)$$

$$\frac{df}{dy} = 2 \cdot \left(x \frac{d\theta}{dV} - y \frac{d\theta}{dU} \right) \textcircled{4}$$

$$\frac{d}{dy} = 2 \left(x \cdot \frac{d}{dV} - y \cdot \frac{d}{dU} \right) \rightarrow \textcircled{5}$$

$$\Rightarrow \frac{d^2 f}{dy^2} = \frac{d}{dy} \left(\frac{df}{dy} \right)$$

$$= 2 \left(x \frac{d}{dV} - y \frac{d}{dU} \right) 2 \left(x \frac{d\theta}{dV} - y \frac{d\theta}{dU} \right)$$

$$= 4 \left[x^2 \frac{d^2 \theta}{dV^2} - xy \frac{d^2 \theta}{dU dV} - xy \frac{d^2 \theta}{dU dV} + y^2 \frac{d^2 \theta}{dU^2} \right] \rightarrow \textcircled{6}$$

adding ③ + ⑥

$$\Rightarrow \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2}$$

$$= 4 \left[x^2 \frac{d^2 \theta}{dU^2} + 2xy \frac{d^2 \theta}{dU dV} + y^2 \frac{d^2 \theta}{dV^2} \right] + 4 \left[x^2 \frac{d^2 \theta}{dV^2} - 2xy \frac{d^2 \theta}{dU dV} + y^2 \frac{d^2 \theta}{dU^2} \right]$$

$$= 4 \left[x^2 \frac{d^2 \theta}{dU^2} + 2xy \frac{d^2 \theta}{dU dV} + y^2 \frac{d^2 \theta}{dV^2} + x^2 \frac{d^2 \theta}{dV^2} - 2xy \frac{d^2 \theta}{dU dV} + y^2 \frac{d^2 \theta}{dU^2} \right]$$

$$= 4 \left[\frac{d^2 \theta}{dU^2} (x^2 + y^2) + \frac{d^2 \theta}{dV^2} (x^2 + y^2) \right]$$

$$= 4(x^2 + y^2) \left(\frac{d^2 \theta}{dU^2} + \frac{d^2 \theta}{dV^2} \right)$$

$$\therefore \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = 4(x^2 + y^2) \left(\frac{d^2 \theta}{dU^2} + \frac{d^2 \theta}{dV^2} \right)$$

23/11/19

Implicit Function:

① If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$. Find the value of $\frac{dz}{dx}$ when $x = y = a$.

$$\text{Given } z = \sqrt{x^2 + y^2}$$

$$x^3 + y^3 + 3axy = 5a^2$$

$$\frac{dz}{dx} = \frac{dz}{dx} \cdot \frac{dx}{dx} + \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$z = \sqrt{x^2 + y^2}$$

$$\frac{dz}{dx} = \frac{dz}{dx} + \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$z = \sqrt{x^2 + y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} (x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{x^2 + y^2}} (y) = \frac{y}{\sqrt{x^2 + y^2}}$$

Given $x^3 + y^3 + 3axy - 5a^2 = 0$

differentiate with respect to 'x'.

$$3x^2 + 3y^2 \frac{dy}{dx} + 3a(y + yx \cdot \frac{dy}{dx}) = 0$$

$$x^2 + y^2 \frac{dy}{dx} + ay + ax \cdot \frac{dy}{dx} = 0$$

$$(y^2 + ax) \frac{dy}{dx} = -(x^2 + ay)$$

$$\frac{dy}{dx} = \frac{-(x^2 + ay)}{y^2 + ax}$$

$$\therefore \frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}} \left(\frac{-(x^2 + ay)}{y^2 + ax} \right)$$

$$= \frac{x}{\sqrt{x^2 + y^2}} - \frac{y(x^2 + ay)}{\sqrt{x^2 + y^2}(y^2 + ax)}$$

$$= \frac{x(y^2 + ax) - y(x^2 + ay)}{\sqrt{x^2 + y^2}(y^2 + ax)}$$

$$= \frac{xy^2 + ax^2 - x^2y - ay^2}{\sqrt{x^2 + y^2}(y^2 + ax)}$$

$$= \frac{(x-a)y^2 + (a-y)x^2}{\sqrt{x^2 + y^2}(y^2 + ax)}$$

$$\frac{dz}{dx} = \frac{(x-a)y^2 - (y-a)x^2}{\sqrt{x^2 + y^2}(y^2 + ax)}$$

at $x=y=a$

$$\frac{dz}{dx} = \frac{(a-a)a^2 - (a-a)a^2}{\sqrt{a^2 + a^2}(a^2 + a^2)}$$

$$= \frac{0-0}{\sqrt{2}a^2(2a^2)}$$

$$\boxed{\frac{dz}{dx} = 0}$$

(2) If $U = x \log(xy)$ where $x^3 + y^3 + 3xy = 1$ find $\frac{dU}{dx}$.

Given $U = x \cdot \log(xy)$

$$U < \frac{y}{x} = x$$

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dx}$$

$$U = x \cdot \log(xy)$$

$$\frac{dU}{dx} = x \cdot \left(\frac{1}{xy}\right)(y) + \log(xy)(1)$$

$$= 1 + \log(xy)$$

$$\frac{dU}{dy} = x \cdot \frac{1}{xy}(x) = \frac{x}{y}$$

Given $x^3 + y^3 + 3xy = 1$

diff. w. r. to 'x'

$$3x^2 + 3y^2 \frac{dy}{dx} + 3\left(x \cdot \frac{dy}{dx} + y(1)\right) = 0$$

$$x^2 + y^2 \frac{dy}{dx} + x \cdot \frac{dy}{dx} + y = 0$$

$$(y^2 + x) \frac{dy}{dx} = -(x^2 + y)$$

$$\frac{dy}{dx} = \frac{-(x^2 + y)}{y^2 + x}$$

$$\frac{dU}{dx} = 1 + \log(xy) + \frac{x}{y} \cdot \left(-\frac{(x^2 + y)}{y^2 + x}\right)$$

$$= 1 + \log(xy) - \frac{x(x^2 + y)}{y(y^2 + x)}$$

$$= \log(xy) + \frac{y^3 + xy - x^3 - xy}{y(y^2 + x)}$$

$$= \log(xy) + \frac{y^3 - x^3}{y^3 + x^3}$$

③ If $z = x^2y$ and $x^2 + xy + y^2 = 1$, find $\frac{dz}{dx}$

Given $z = x^2y$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

① $z = x^2y$

$$\frac{\partial z}{\partial x} = y(2x) = 2xy$$

$$\frac{\partial z}{\partial y} = x^2(1) = x^2$$

Given $x^2 + xy + y^2 = 1$

diff. w. r. to 'x'

$$2x + x \cdot \frac{dy}{dx} + y(1) + 2y = 0$$

$$2x + x \cdot \frac{dy}{dx} + 3y = 0$$

$$x \cdot \frac{dy}{dx} = -(2x + 3y)$$

$$\frac{dy}{dx} = \frac{-(2x + 3y)}{x}$$

$$\frac{dz}{dx} = 2xy + x \cdot \frac{-(2x + 3y)}{x}$$

$$= 2xy - x(2x + 3y)$$

$$= 2xy - 2x^2 - 3xy = -2x^2 - xy$$

$$= -(2x^2 + xy)$$

⑤ If $xy = y^x$ then find $\frac{dy}{dx}$

Given $xy = y^x$

$$xy - y^x = 0$$

$$f(x, y) = xy - y^x \rightarrow \text{①}$$

$$\therefore \frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

differentiate Eqn ① w.r. to 'x' partially.

$$\Rightarrow \frac{\partial f}{\partial x} = y \cdot x^{y-1} - y^x \cdot \log y$$

diff. w.r. to 'y' Partially.

$$\Rightarrow \frac{\partial f}{\partial y} = xy \cdot \log x - x \cdot y^{x-1}$$

$$\therefore \frac{dy}{dx} = \frac{y \cdot x^{y-1} - y^x \cdot \log y}{xy \cdot \log x - x \cdot y^{x-1}}$$

⑥ Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$

Given $(\cos x)^y = (\sin y)^x$

$$(\cos x)^y - (\sin y)^x = 0$$

$$f(x, y) = (\cos x)^y - (\sin y)^x \rightarrow \text{②}$$

$$\frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

diff. equ'n w. r. to 'x' partially,

$$\Rightarrow \frac{df}{dx} = y(\cos x)^{y-1} (-\sin x) - \sin y^x \cdot \log \sin y (\cos y)^x \cos$$

$$= -y \sin x (\cos x)^{y-1} - \cos y / (\sin y)^x \cdot \log \sin y$$

diff. equ'n w. respect to 'y' partially.

$$\Rightarrow \frac{df}{dy} = (\cos x)^y \cdot \log \cos x (\cos x)^0 - x (\sin y)^{x-1} \cos y$$

$$= + \sin x (\cos x)^y \cdot \log (\cos x) - x \cos y (\sin y)^{x-1}$$

$$\therefore \frac{dy}{dx} = \frac{(-y \sin x (\cos x)^{y-1} - \cos y (\sin y)^x \cdot \log \sin y)}{(\sin x (\cos x)^y \cdot \log \cos x + x \cos y (\sin y)^{x-1})}$$

$$= \frac{(-y \sin x (\cos x)^{y-1} - \cos y (\sin y)^x \cdot \log \sin y)}{\sin x (\cos x)^y \cdot \log \cos x + x \cos y (\sin y)^{x-1}}$$

~~$\frac{dy}{dx} \neq$~~
equ'n

diff. " w. r. to 'x' partially

$$\Rightarrow \frac{df}{dx} = y(\cos x)^{y-1} (-\sin x) - (\sin y)^x \cdot \log \sin y$$

diff. equ'n w. r. to 'y' partially.

$$\Rightarrow \frac{df}{dy} = (\cos x)^y \cdot \log (\cos x) - x (\sin y)^{x-1} \cdot \cos y$$

$$\therefore \frac{dy}{dx} = \frac{(-y \sin x (\cos x)^{y-1} - (\sin y)^x \cdot \log (\sin y))}{(\cos x)^y \cdot \log (\cos x) - x \frac{(\sin y)^x}{\sin y} \cdot \cos y}$$

$$= \frac{y \cdot \tan x (\cos x)^y + (\cos x)^y \cdot \log \sin y}{(\cos x)^y \log (\cos x) - x (\sin y)^x \cot y}$$

$$= \frac{(\cos x)^y [y \tan x + \log (\sin y)]}{(\cos x)^y [\log \cos x - x \cot y]}$$

$$= \frac{y \tan x + \log (\sin y)}{\log (\cos x) - x \cot y}$$

④ If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$. Find $\frac{dy}{dx}$

Given that $x^3 + 3x^2y + 6xy^2 + y^3 = 1$

$$x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$$

$$f(x, y) = x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$$

$$\frac{dy}{dx} = - \frac{\frac{df}{dx}}{\frac{df}{dy}}$$

diff. equⁿ w. a. to 'x' partially

$$\Rightarrow \frac{df}{dx} = 3x^2 + 3y(2x) + 6y^2(1) + 0 - 0$$

$$= 3x^2 + 6xy + 6y^2$$

diff. equⁿ w. a. to 'y' partially

$$\Rightarrow \frac{df}{dy} = 0 + 3x^2(1) + 6x(2y) + 3y^2 - 0$$

$$= 3x^2 + 12xy + 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2 + 6xy + 6y^2)}{3x^2 + 12xy + 3y^2}$$

$$= \frac{-3(x^2 + 2xy + 2y^2)}{3(x^2 + 4xy + y^2)}$$

$$= \frac{-(x^2 + 2xy + 2y^2)}{x^2 + 4xy + y^2}$$

⑥ If $x^3 + y^3 - 3axy = 0$. Find $\frac{dy}{dx}$

Given that $x^3 + y^3 - 3axy = 0$

$$f(x, y) = x^3 + y^3 - 3axy = 0$$

diff. equⁿ w. a. to 'x' partially

$$\frac{df}{dx} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay$$

diff. w. a. to 'y' partially

$$\frac{df}{dy} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2 - 3ay)}{(3y^2 - 3ax)} = \frac{-(x^2 - ay)}{y^2 - ax}$$

⑦ Prove that Find $\frac{dy}{dx}$ If $y^3 - 3ax^2 + x^3 = 0$.

Sol

Given that $y^3 - 3ax^2 + x^3 = 0$

$$f(x, y) = y^3 - 3ax^2 + x^3 \rightarrow \text{①}$$

diff. equⁿ w. r. to 'x' partially.

$$\frac{df}{dx} = 0 - 3a(2x) + 3x^2 = 3x^2 - 6ax$$

diff. equⁿ w. r. to 'y' partially.

$$\frac{df}{dy} = 3y^2 - 0 + 0 = 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2 - 6ax)}{3y^2} = \frac{-3(x^2 - 2ax)}{3y^2} = \frac{2ax - x^2}{y^2}$$

⑧ Find $\frac{dy}{dx}$ when $xy + y^x = c$.

Given that $xy + y^x = c$

$$xy + y^x - c = 0$$

$$f(x, y) = xy + y^x - c \rightarrow \text{①}$$

diff. equⁿ w. r. to 'x' partially.

$$\frac{df}{dx} = y \cdot x^{y-1} + y^x \log y - 0 = y \cdot x^{y-1} + y^x \log y$$

diff. equⁿ w. r. to 'y' partially

$$\frac{df}{dy} = x \cdot \log x + x \cdot y^{x-1} - 0 = x \cdot \log x + x \cdot y^{x-1}$$

$$\therefore \frac{dy}{dx} = \frac{-(y \cdot x^{y-1} + y^x \log y)}{x \cdot \log x + x \cdot y^{x-1}}$$

26/11/19 Tuesday Taylor's (Expansion) Theorem:

* Expand the following functions.

① $f(x, y) = e^x \sin y$

By Maclaurin's expansion,

$$f(x, y) = f(0, 0) + [x \cdot f_x(0, 0) + y \cdot f_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) + y^2 f_{yy}(0, 0) + 2xy f_{xy}(0, 0)] + \dots$$

Now, $f(x, y) = e^x \sin y$

$\Rightarrow f(0, 0) = e^0 \cdot \sin(0) = (1)(0) = 0.$

$\Rightarrow f_x = \frac{\partial f}{\partial x} = \sin y \cdot e^x \Rightarrow f_x(0, 0) = \sin(0) e^0 = 0.$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = e^x \cdot \cos y \Rightarrow f_y(0, 0) = e^0 \cdot \cos(0) = 1$

$\Rightarrow f_{xx} = \frac{\partial^2 f}{\partial x^2} = \sin y \cdot e^x \Rightarrow f_{xx}(0, 0) = \sin(0) e^0 = 0$

$\Rightarrow f_{yy} = \frac{\partial^2 f}{\partial y^2} = e^x \cdot (-\sin y) \Rightarrow f_{yy}(0, 0) = e^0 \sin(0) = 0.$

$\Rightarrow f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^x \cdot \cos y \Rightarrow f_{xy}(0, 0) = e^0 \cdot \cos(0) = 1$

$\therefore e^x \sin y = 0 + [x(0) + y(1)] + \frac{1}{2!} [x^2(0) + y^2(0) + 2xy(1)] + \dots$

$= 0 + 0 + y + 0 + 0 + \frac{1}{2!} (2xy) + \dots$

$= y + xy + \dots$

② $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ up to third degree terms. Hence compute $f(1.1, 0.9)$ approximately.

By Taylor's expansion at the (a, b) is

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \frac{1}{2!} [(x-a)^2 f_{xx}(a, b) + (y-b)^2 f_{yy}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)] + \dots$$

$$+ \frac{1}{3!} [(x-a)^3 f_{xxx}(a, b) + (y-b)^3 f_{yyy}(a, b) + 3(x-a)(y-b) f_{xxy}(a, b) + 3(x-a)(y-b) f_{xyy}(a, b)] + \dots$$

at $(1, 1)$.

$$f(x, y) = f(1, 1) + [(x-1)f_x(1, 1) + (y-1)f_y(1, 1)] + \frac{1}{2!} [(x-1)^2 f_{xx}(1, 1) + (y-1)^2 f_{yy}(1, 1) + 2(x-1)(y-1) f_{xy}(1, 1)] + \frac{1}{3!} [(x-1)^3 f_{xxx}(1, 1) + (y-1)^3 f_{yyy}(1, 1) + 3(x-1)(y-1) f_{xxy}(1, 1) + 3(x-1)(y-1) f_{xyy}(1, 1)] + \dots \rightarrow \textcircled{1}$$

We have $f(x,y) = \tan^{-1}(y/x)$

$\Rightarrow f(1,1) = \tan^{-1}(1) = \tan^{-1}(1) = \pi/4$

$f_x = \frac{df}{dx} = \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{y^2+x^2} \Rightarrow f_x(1,1) = \frac{-1}{1+1} = -\frac{1}{2}$

$f_y = \frac{df}{dy} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{x}{y^2+x^2} \Rightarrow f_y(1,1) = \frac{1}{1+1} = \frac{1}{2}$

$f_{xx} = \frac{d^2f}{dx^2} = (-y) \frac{-1}{(x^2+y^2)^2} (2x) \Rightarrow f_{xx}(1,1) = -1 \frac{-1}{(1+1)^2} (2(1))$
 $= \frac{2}{2} = 1$

$f_{yy} = \frac{d^2f}{dy^2} = x \frac{-1}{(x^2+y^2)^2} (2y) = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{-2}{(1+1)^2} = -\frac{1}{2}$

$f_{xy} = \frac{d^2f}{dx dy} = \frac{1}{1+(y/x)^2} \left(\frac{y}{x^2} \right) (y^2+x^2)(-1) - \left(\frac{y}{x} \right) (2y+x) = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$
 $= \frac{-x^2+y^2+2xy}{(x^2+y^2)^2} = \frac{-(x-y)^2}{(x^2+y^2)^2}$

$\Rightarrow f_{xy}(1,1) = \frac{-(1-1)^2}{(1+1)^2} = 0$

$\Rightarrow f_{xy}(1,1) = \frac{-1-1+2}{(1+1)^2} = 0$

$f_{xxx} = \frac{(x^2+y^2)^2 (2y)(-1) - 2xy \cdot 2(x^2+y^2)(2x)}{[(x^2+y^2)^2]^2}$

$= \frac{2y(x^2+y^2)^2 - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$

$\Rightarrow f_{xxx}(1,1) = \frac{2(1)(1+1)^2 - 8(1)(1)(1+1)}{(1+1)^4} = \frac{8-16}{16} = \frac{-8}{16} = -\frac{1}{2}$

$f_{yyy} = \frac{(x^2+y^2)^2 (2x)(1) + 2xy \cdot 2(x^2+y^2)(0+2y)}{(x^2+y^2)^4}$

$= \frac{-2x(x^2+y^2)^2 + 8xy^2(x^2+y^2)}{(x^2+y^2)^4}$

$\Rightarrow f_{yyy}(1,1) = \frac{-2(1) + 8(1)(1)}{16} = \frac{-2+8}{16} = \frac{6}{16} = \frac{3}{8}$

$f_{xxy} = 2x \left[\frac{(x^2+y^2)^2 (1) - y \cdot 2(x^2+y^2) (2y)}{[(x^2+y^2)^2]^2} \right]$

$= 2x \left[\frac{x^2+y^2-4y^2}{(x^2+y^2)^3} \right]$

$$\Rightarrow f_{xy}(1,1) = \frac{d}{dy} \left[\frac{y-4}{y} \right] = \frac{-2}{y} = -\frac{2}{1} = -2.$$

$$\text{Hence } f_{xy}(1,1) = \frac{1}{2}.$$

from (1),

$$\tan^{-1}(y/x) = \pi/4 + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) \right] + \frac{1}{2!} \left[(x-1)^2 \left(-\frac{1}{2}\right) + (y-1)^2 \left(\frac{1}{2}\right) + 2(x-1)(y-1) \left(-\frac{1}{2}\right) \right]$$

$$f(x,y) = \frac{\pi}{4} + \frac{1}{2} \left[-(x-1) + (y-1) \right] + \frac{1}{2!} \left[\frac{1}{2} \left[(x-1)^2 - (y-1)^2 \right] + \frac{1}{3!} \left[-(x-1)^3 + (y-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 \right] \right]$$

$$= \frac{\pi}{4} + \frac{1}{2} \left[-(x-1) + (y-1) \right] + \frac{1}{4} \left[(x-1)^2 - (y-1)^2 \right] + \frac{1}{24} \left[-(x-1)^3 + (y-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 \right] + \dots$$

$$f(1.1, 0.9) = \frac{\pi}{4} + \frac{1}{2} \left[-(1.1-1) + (0.9-1) \right] + \frac{1}{4} \left[(1.1-1)^2 - (0.9-1)^2 \right] + \frac{1}{24} \left[-(1.1-1)^3 + (0.9-1)^3 + 3(1.1-1)^2(0.9-1) - 3(1.1-1)(0.9-1)^2 \right]$$

$$= \frac{3.14}{4} + \frac{1}{2} \left[-0.1 \right] + \frac{1}{4} \left[0.01 \right] + \frac{1}{24} \left[-0.001 + 0.001 + 3(0.01)(-0.1) - 3(0.1)(0.01) \right]$$

$$= 0.785 - 0.1 + 0.00025 + \frac{1}{24} \left[0.001 - 0.001 - 0.003 - 0.003 \right] + 3(0.01)(-0.1) - 3(0.1)(0.01)$$

$$= 0.68533.$$

④ $f(x,y) = e^x \log(1+x)$

sol:- $f(x,y) = e^x \log(1+x)$

$$f(x,y) = f(0,0) + \left[x \cdot f_x(0,0) + y \cdot f_y(0,0) \right] + \frac{1}{2!} \left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0) \right]$$

We have,

$$f(x,y) = e^y \log(1+x) \Rightarrow f(0,0) = e^0 \cdot [\log(1)] = 0.$$

$$f_x = \frac{df}{dx} = e^y \cdot \frac{1}{1+x} \Rightarrow f_x(0,0) = e^0 \cdot \frac{1}{1+0} = 1.$$

$$f_y = \frac{df}{dy} = \log(1+x) e^y \Rightarrow f_y(0,0) = 0.$$

$$f_{xx} = \frac{d^2f}{dx^2} = e^y \cdot \frac{-1}{(1+x)^2} \Rightarrow f_{xx}(0,0) = e^0 \cdot \frac{-1}{(1+0)^2} = -1.$$

$$f_{xy} = \frac{d^2f}{dx dy} = \frac{1}{1+x} e^y \Rightarrow f_{xy}(0,0) = \frac{1}{1+0} e^0 = 1.$$

$$f_{yy} = \frac{d^2f}{dy^2} = \log(1+x) e^y \Rightarrow f_{yy}(0,0) = 0.$$

$$e^y \log(1+x) = 0 + [x(1) + y(0)] + \frac{1}{2!} [x^2(-1) + 2xy(1) + y^2(0)] + \dots$$

$$= x + \frac{1}{2} (-x^2 + 2xy) + \dots$$

$$= x - \frac{x^2}{2} + xy + \dots$$

③ $f(x, y) = e^x \log(1+y)$

Given $f(x, y) = e^x \log(1+y)$

By Maclaurin's expansion

$$f(x, y) = f(0, 0) + [x \cdot f_x(0, 0) + y \cdot f_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) + y^2 f_{yy}(0, 0) + 2xy f_{xy}(0, 0)] + \dots \rightarrow \textcircled{1}$$

$$f(x, y) = e^x \log(1+y) \Rightarrow f(0, 0) = e^0 \log(1+0) = 0.$$

$$f_x = \frac{df}{dx} = \log(1+y) e^x \Rightarrow f_x(0, 0) = \log(1+0) e^0 = 0.$$

$$f_y = \frac{df}{dy} = e^x \cdot \frac{1}{1+y} \Rightarrow f_y(0, 0) = e^0 \frac{1}{1+0} = 1.$$

$$f_{xx} = \frac{d^2 f}{dx^2} = \log(1+y) e^x \Rightarrow f_{xx}(0, 0) = \log(1+0) e^0 = 0.$$

$$f_{yy} = \frac{d^2 f}{dy^2} = e^x \cdot \frac{-1}{(1+y)^2} \Rightarrow f_{yy}(0, 0) = e^0 \frac{-1}{(1+0)^2} = -1$$

$$f_{xy} = \frac{d^2 f}{dx dy} = e^x \cdot \frac{1}{1+y} \Rightarrow f_{xy}(0, 0) = e^0 \frac{1}{1+0} = 1$$

from $\textcircled{1}$,

$$e^x \cdot \log(1+y) = 0 + [x(0) + y(1)] + \frac{1}{2!} [x^2(0) + y^2(-1) + 2xy(1)] + \dots$$

$$= y + \frac{1}{2} (-y^2 + 2xy) + \dots$$

$$= y - \frac{y^2}{2} + xy + \dots$$

④ Expand $x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$ using Taylor's theorem.

By Taylor's expansion,

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!} [(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \dots$$

$$= f(1, -2) + [(x-1) f_x(1, -2) + (y+2) f_y(1, -2)] + \frac{1}{2!} [(x-1)^2 f_{xx}(1, -2) + 2(x-1)(y+2) f_{xy}(1, -2) + (y+2)^2 f_{yy}(1, -2)] + \dots$$

$\rightarrow \textcircled{1}$

We have $f(x,y) = x^2y + 3y - 2 \Rightarrow f(1,-2) = -2 - 6 - 2 = -10$

$f_x = \frac{\partial f}{\partial x} = 2y(1) + 0 - 0 \Rightarrow f_x(1,-2) = -4$

$f_y = \frac{\partial f}{\partial y} = x^2(1) + 3(1) - 0 \Rightarrow f_y(1,-2) = 1 + 3 = 4$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2y(1) \Rightarrow f_{xx}(1,-2) = -4$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x(1) \Rightarrow f_{xy}(1,-2) = 2$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 0 + 0 \Rightarrow f_{yy}(1,-2) = 0$

from (1),

$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2!} [(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \dots$

$= -10 + f_1$

$= -10 - 4[(x-1) - (y+2)] + \frac{1}{2} [(x-1)^2 - 2(x-1)(y+2)] + \dots$

$= -10 - 4[(x-1) - (y+2)] - 2[(x-1)^2 - 2(x-1)(y+2)] + \dots$

(8) Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$

and hence deduce that $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$

By Maclaurin's expansion,

$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots$

We have $f(x) = \log(1+e^x) \Rightarrow f(0) = \log(1+e^0) = \log 2$

$f'(x) = \frac{1}{1+e^x} \cdot e^x \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$

$f''(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} \Rightarrow f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{4}$

$f'''(x) = \frac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4} = \frac{(1+e^x)[(1+e^x)e^x - 2e^{2x}]}{(1+e^x)^4}$

$= \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3} \Rightarrow f'''(x) =$

$\Rightarrow f'''(0) = \frac{e^0 - e^0}{(1+e^0)^3} = 0$

$f^{(4)}(x) = \frac{(1+e^x)^3 [e^x - e^{2x}(2)] - (e^x - e^{2x}) 3(1+e^x)^2 e^x}{(1+e^x)^6}$

$$f^{IV}(x) = \frac{(1+e^x)^2 [1+e^x(e^x-2e^{2x}) - 3e^x(e^x-e^{2x})]}{(1+e^x)^4}$$

$$\Rightarrow f^{IV}(0) = \frac{(1+e^0)(e^0-2e^0) - 3e^0(e^0-e^0)}{(1+e^0)^4}$$

$$= \frac{2(1-2) - 3(1)(1-1)}{(1+1)^4} = \frac{-2-0}{16} = \frac{-2}{16} = -\frac{1}{8}$$

$$\log(1+e^x) = \log 2 + x \cdot \frac{1}{2} + \frac{x^2}{2!} \cdot \frac{1}{4} + \frac{x^3}{3!} (0) + \frac{x^4}{4!} \left(-\frac{1}{8}\right) + \dots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$$

diff. w. x. to 'x'

$$\frac{1}{1+e^x} \cdot e^x = 0 + \frac{1}{2} + \frac{1}{8}(2x) - \frac{4x^3}{192} + \dots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$$

⑤ $f(x, y) = e^{xy}$ in powers of $(x-1)$ and $(y-1)$.

By Taylor's expansion,

$$f(x, y) = f(1, 1) + [(x-1)f_x(1, 1) + (y-1)f_y(1, 1)] + \frac{1}{2!} [(x-1)^2 f_{xx}(1, 1) + (y-1)^2 f_{yy}(1, 1) + 2(x-1)(y-1)f_{xy}(1, 1)] + \dots$$

$$\text{We have } f(x, y) = e^{xy} \Rightarrow f(1, 1) = e^{(1)(1)} = e$$

$$f_x = \frac{df}{dx} = e^{xy}(y) \Rightarrow f_x(1, 1) = e^{(1)(1)}(1) = e$$

$$f_y = \frac{df}{dy} = e^{xy}(x) \Rightarrow f_y(1, 1) = e^{(1)(1)}(1) = e$$

$$f_{xx} = \frac{d^2f}{dx^2} = y \cdot e^{xy}(y) \Rightarrow f_{xx}(1, 1) = (1) e^{(1)(1)}(1) = e$$

$$f_{yy} = \frac{d^2f}{dy^2} = x \cdot e^{xy}(x) \Rightarrow f_{yy}(1, 1) = (1) e^{(1)(1)}(1) = e$$

$$f_{xy} = \frac{d^2f}{dx dy} = e^{xy}(1) + y \cdot e^{xy}(x) \Rightarrow f_{xy}(1, 1) = e + e = 2e$$

$$e^{xy} = e + [(x-1)e + (y-1)e] + \frac{1}{2!} [(x-1)^2 e + (y-1)^2 e + 2(x-1)(y-1)2e] + \dots$$

$$= e + e[(x-1) + (y-1)] + \frac{e}{2!} [(x-1)^2 + (y-1)^2 + 4(x-1)(y-1)] + \dots$$

⑥ $f(x, y) = e^x \cos y$ about $(1, \pi/4)$.

By Taylor's expansion,

$$f(x, y) = f(1, \pi/4) + [(x-1)f_x(1, \pi/4) + (y-\pi/4)f_y(1, \pi/4)] + \frac{1}{2!} [(x-1)^2 f_{xx}(1, \pi/4) + 2(x-1)(y-\pi/4)f_{xy}(1, \pi/4) + (y-\pi/4)^2 f_{yy}(1, \pi/4)] + \dots$$

We have $f(x, y) = e^x \cos y \Rightarrow f(1, \pi/4) = e^1 \cos \pi/4 = \frac{e}{\sqrt{2}}$

$$f_x = \frac{df}{dx} = \cos y \cdot e^x \Rightarrow f_x(1, \pi/4) = \cos \pi/4 \cdot e^1 = \frac{e}{\sqrt{2}}$$

$$f_y = \frac{df}{dy} = e^x \cdot (-\sin y) \Rightarrow f_y(1, \pi/4) = -e^1 \sin \pi/4 = -\frac{e}{\sqrt{2}}$$

$$f_{xx} = \cos y \cdot e^x \Rightarrow f_{xx}(1, \pi/4) = \cos \pi/4 \cdot e^1 = \frac{e}{\sqrt{2}}$$

$$f_{xy} = e^x (-\sin y) \Rightarrow f_{xy}(1, \pi/4) = -e^1 \sin \pi/4 = -\frac{e}{\sqrt{2}}$$

$$f_{yy} = -e^x \cos y \Rightarrow f_{yy}(1, \pi/4) = -e^1 \cos \pi/4 = -\frac{e}{\sqrt{2}}$$

$$\begin{aligned} e^x \cos y &= \frac{e}{\sqrt{2}} + [(x-1)\frac{e}{\sqrt{2}} + (y-\pi/4)(-\frac{e}{\sqrt{2}})] + \frac{1}{2!} [(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)(y-\pi/4)(-\frac{e}{\sqrt{2}}) + (y-\pi/4)^2 (-\frac{e}{\sqrt{2}})] + \dots \\ &= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}} [(x-1) - (y-\pi/4)] + \frac{1}{2!} \frac{e}{\sqrt{2}} [(x-1)^2 - 2(x-1)(y-\pi/4) - (y-\pi/4)^2] + \dots \\ &= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}} [(x-1) - (y-\pi/4)] + \frac{e}{2\sqrt{2}} [(x-1)^2 - 2(x-1)(y-\pi/4) - (y-\pi/4)^2] + \dots \end{aligned}$$

⑦ $f(x, y) = \sin xy$ in powers of $(x-1)$ and $(y-\pi/2)$ up to second degree terms.

By Taylor's expansion,

$$f(x, y) = f(1, \pi/2) + [(x-1)f_x(1, \pi/2) + (y-\pi/2)f_y(1, \pi/2)] + \frac{1}{2!} [(x-1)^2 f_{xx}(1, \pi/2) + 2(x-1)(y-\pi/2)f_{xy}(1, \pi/2) + (y-\pi/2)^2 f_{yy}(1, \pi/2)] + \dots$$

We have,

$$f(x, y) = \sin xy \Rightarrow f(1, \pi/2) = \sin \pi/2 = 1$$

$$f_x = \frac{df}{dx} = \cos xy (y) \Rightarrow f_x(1, \pi/2) = (\pi/2) \cos \pi/2 = 0$$

$$f_y = \frac{df}{dy} = \cos xy (x) \Rightarrow f_y(1, \pi/2) = (1) \cos \pi/2 = 0$$

$$f_{xx} = y \cdot (-\sin xy) (y) \Rightarrow f_{xx}(1, \pi/2) = \pi/2 \cdot \pi/2 (-\sin \pi/2) = \frac{\pi^2}{4} (-1) = -\frac{\pi^2}{4}$$

$$f_{xy} = \frac{d^2 f}{dx dy} = y \cdot (-\sin xy)(x) + \cos xy(1)$$

$$\Rightarrow f_{xy}(1, \pi/2) = \pi/2 - \sin \pi/2(1) + \cos \pi/2$$

$$= -\pi/2(1) + 0 = -\pi/2$$

$$f_{yy} = \frac{d^2 f}{dy^2} = x \cdot (-\sin xy)(x) = (1) - \sin \pi/2(1) = -1$$

$$\sin xy = 1 + [(\tilde{x}-1)0 + (y-\pi/2)0] + \frac{1}{2!}[(\tilde{x}-1)(-\pi/2) + 2(\tilde{x}-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)] + \dots$$

$$= 1 + [0+0] + \frac{1}{2}[(\tilde{x}-1)(-\pi/2) + 0 + (y-\pi/2)^2(-1)] + \dots$$

$$\sin xy = 1 + \frac{1}{2}[(\tilde{x}-1)(-\pi/2) + (y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)] + \dots$$

28/11/19

Thursday Jacobian:

② If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$. show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$

sol:-

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\begin{matrix} x \\ y \\ z \end{matrix} \rightarrow r \theta \phi$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi(1)$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \phi$$

$$\frac{\partial z}{\partial r} = \cos \theta$$

$$\frac{\partial x}{\partial \theta} = r \cos \phi \cos \theta$$

$$\frac{\partial y}{\partial \theta} = r \sin \phi \cos \theta$$

$$\frac{\partial z}{\partial \theta} = r(-\sin \theta)$$

$$\frac{\partial x}{\partial \phi} = r \sin \theta (-\sin \phi)$$

$$\frac{\partial y}{\partial \phi} = r \sin \theta \cos \phi$$

$$\frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \overset{(+)}{\sin \theta \cos \phi} & \overset{(-)}{r \cos \theta \cos \phi} & \overset{(+)}{-r \sin \theta \sin \phi} \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$\begin{aligned}
 &= \sin \theta \cdot \cos \phi [0 + r^2 \sin^2 \theta \cdot \cos \phi] - r \cos \theta \cdot \cos \phi [0 - r \sin \theta \cos \theta \cos \phi] \\
 &\quad - r \sin \theta \cdot \sin \phi [-r \sin^2 \theta \cdot \sin \phi - r \cos^2 \theta \cdot \sin \phi] \\
 &= r^2 \sin^3 \theta \cdot \cos^2 \phi + r^2 \sin \theta \cdot \cos^2 \theta \cdot \cos^2 \phi - r \sin \theta \cdot \sin \phi \\
 &\quad \cdot (-r \sin \theta) [\sin^2 \theta + \cos^2 \theta] \\
 &= r^2 \sin^3 \theta \cdot \cos^2 \phi + r^2 \sin \theta \cdot \cos^2 \theta \cdot \cos^2 \phi + r^2 \sin \theta \cdot \sin^2 \phi \\
 &= r^2 \sin \theta \cdot \cos^2 \phi [\sin^2 \theta + \cos^2 \theta] + r^2 \sin \theta \cdot \sin^2 \phi \\
 &= r^2 \sin \theta \cdot \cos^2 \phi (1) + r^2 \sin \theta \cdot \sin^2 \phi \\
 &= r^2 \sin \theta [\cos^2 \phi + \sin^2 \phi] \\
 &= r^2 \sin \theta.
 \end{aligned}$$

③ If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$, show that $\frac{d(uvw)}{d(xyz)} = 0$.

Sol: $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$ $uvw < \frac{1}{x-y}$ (or)

$u > xyz$

$$\frac{d(uvw)}{d(xyz)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$u = \frac{x}{y-z}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y-z}$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{-1}{(y-z)^2}$$

$$\frac{\partial u}{\partial z} = x \cdot \frac{-1}{(y-z)^2} (-1) = \frac{x}{(y-z)^2}$$

$$v = \frac{y}{z-x}$$

$$\frac{\partial v}{\partial x} = y \cdot \frac{-1}{(z-x)^2} (-1) = \frac{y}{(z-x)^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{z-x}$$

$$\frac{\partial v}{\partial z} = y \cdot \frac{-1}{(z-x)^2}$$

$$w = \frac{z}{x-y}$$

$$\frac{\partial w}{\partial x} = z \cdot \frac{-1}{(x-y)^2}$$

$$\frac{\partial w}{\partial y} = z \cdot \frac{-1}{(x-y)^2} (-1) = \frac{z}{(x-y)^2}$$

$$\frac{\partial w}{\partial z} = \frac{1}{x-y}$$

$$\frac{d(uvw)}{d(xyz)} = \begin{vmatrix} \frac{1}{y-z} & \frac{-x}{(y-z)^2} & \frac{x}{(y-z)^2} \\ \frac{y}{(z-x)^2} & \frac{1}{z-x} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{x-y} \end{vmatrix}$$

$$\begin{aligned}
 &= \frac{1}{y-z} \left[\frac{1}{(x-y)(z-x)} + \frac{yz}{(x-y)^2(z-x)^2} \right] + \frac{x}{(y-z)^2} \left[\frac{y}{(x-y)(z-x)^2} \right. \\
 &\quad \left. - \frac{zy}{(x-y)^2(z-x)^2} \right] + \frac{x}{(y-z)^2} \left[\frac{yz}{(z-x)^2(x-y)^2} + \frac{-z}{(x-y)^2(z-x)} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{y-z} \frac{1}{x-y} \frac{1}{z-x} \left[1 + \frac{yz}{(x-y)(z-x)} \right] + \frac{xy}{(x-y)(y-z)(z-x)^2} \left[1 - \frac{z}{x-y} \right] \\
 &\quad + \frac{xz}{(y-z)^2(z-x)(x-y)} \left[\frac{y}{z-x} + 1 \right] \\
 &= \frac{1}{(x-y)(y-z)(z-x)} \left[\frac{(x-y)(z-x) + yz}{(x-y)(z-x)} \right] + \frac{xy}{(x-y)(y-z)(z-x)^2} \left[\frac{x-y-z}{x-y} \right] \\
 &\quad + \frac{xz}{(y-z)^2(z-x)(x-y)} \left[\frac{y+z-x}{z-x} \right] \\
 &= \frac{1}{(x-y)(y-z)(z-x)^2} [xz - x^2 - yz + xy + yz] + \frac{xy}{(x-y)^2(y-z)(z-x)^2} (x-y-z) \\
 &\quad + \frac{xz}{(y-z)^2(z-x)(x-y)^2} [y+z-x] \\
 &= \frac{1}{(x-y)(y-z)(z-x)^2} \left[(y-z)(xz - x^2 + xy) + xy(x-y-z) + xz(y+z-x) \right] \\
 &= \frac{1}{(x-y)(y-z)(z-x)^2} [xy^2 - x^2y + xy^2 - xz^2 + zx^2 - xy^2 + x^2y - xy^2 - xy^2 \\
 &\quad + xy^2 + xz^2 - x^2z] \\
 &= \frac{1}{(x-y)(y-z)(z-x)^2} (0) \\
 &= 0.
 \end{aligned}$$

① If $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$, evaluate $\frac{d(r, \theta)}{d(x, y)}$.

Sol:-

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(y/x)$$

$$\tan \theta = y/x$$

$$r, \theta < \frac{y}{x}$$

$$r, \theta > \frac{y}{x}$$

$$\frac{d(r, \theta)}{d(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} (x)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot \theta \left(-\frac{1}{x^2} \right)$$

$$\frac{\partial r}{\partial y} = \frac{1}{\sqrt{x^2 + y^2}} (y)$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$\frac{d(r, \theta)}{d(x, y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix}$$

$$= \frac{x^2}{\sqrt{x^2+y^2}(x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2}(x^2+y^2)}$$

$$= \frac{x^2+y^2}{\sqrt{x^2+y^2}(x^2+y^2)}$$

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{\sqrt{x^2+y^2}}$$

② If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(xy z)}{\partial(uvw)} = \frac{1}{4}$.

$$\frac{\partial(xy z)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$uvw \begin{matrix} \swarrow x \\ \searrow y \\ \nwarrow z \end{matrix}$$

$$\begin{array}{l} \frac{\partial u}{\partial x} = yz \left(\frac{1}{x^2} \right) \quad \frac{\partial v}{\partial x} = \frac{z}{y} \quad \frac{\partial w}{\partial x} = \frac{y}{z} \\ \frac{\partial u}{\partial y} = \frac{z}{x} \quad \frac{\partial v}{\partial y} = zx \left(\frac{-1}{y^2} \right) \quad \frac{\partial w}{\partial y} = \frac{x}{z} \\ \frac{\partial u}{\partial z} = \frac{y}{x} \quad \frac{\partial v}{\partial z} = \frac{x}{y} \quad \frac{\partial w}{\partial z} = xy \left(\frac{-1}{z^2} \right) \end{array}$$

$$\frac{\partial(xy z)}{\partial(uvw)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{y} & \frac{y}{z} \\ \frac{z}{x} & -\frac{zx}{y^2} & \frac{x}{z} \\ \frac{y}{x} & \frac{x}{y} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{xyz}{x^2} \left[\frac{xy z}{y^2 z^2} - \frac{x^2}{yz} \right]$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{zx}{x^2} & \frac{xy}{x^2} \\ \frac{zy}{y^2} & -\frac{zx}{y^2} & \frac{xy}{y^2} \\ \frac{yz}{z^2} & \frac{xz}{z^2} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{(yz)(zx)(xy)}{x^2 y^2 z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{x^2 y^2 z^2}{x^2 y^2 z^2} [-1(-1) - 1(-1) + 1(1+1)]$$

$$= 0 + 2 + 2 = 4 \Rightarrow \frac{\partial(xy z)}{\partial(uvw)} = \frac{1}{4}$$

We know that,

$$\frac{d(uvw)}{d(xyz)} \cdot \frac{d(xyz)}{d(uvw)} = 1$$

$$4. \frac{d(xyz)}{d(uvw)} = 1$$

$$\boxed{\frac{d(xyz)}{d(uvw)} = 1/4}$$

⑭ $u = x + y + z$; $uv = y + z$; $uvw = z$ show that $\frac{d(xyz)}{d(uvw)} = u^2v$.

$$u = x + y + z$$

$$uv = y + z$$

$$uvw = z$$

$$u = x + uv$$

$$uv = y + uvw$$

$$z = uvw$$

$$x = u - uv$$

$$y = uv - uvw$$

$$x \ y \ z \begin{cases} u \\ v \\ w \end{cases}$$

$$\frac{d(xyz)}{d(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$x = u - uv$$

$$y = uv - uvw$$

$$z = uvw$$

$$\frac{\partial x}{\partial u} = 1 - v$$

$$\frac{\partial y}{\partial u} = v - vw$$

$$\frac{\partial z}{\partial u} = vw$$

$$\frac{\partial x}{\partial v} = 0 - u$$

$$\frac{\partial y}{\partial v} = u - uw$$

$$\frac{\partial z}{\partial v} = uw$$

$$\frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial w} = 0 - uv$$

$$\frac{\partial z}{\partial w} = uv$$

$$\frac{d(xyz)}{d(uvw)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[(u-uw)uv + u^2vw] + u[(v-vw)uv + uv^2w] + 0 -$$

$$= (1-v)[u^2v - u^2vw + u^2vw] + u[uv^2 - uv^2w + uv^2w]$$

$$= u^2v - u^2v^2w + u^2v^2w$$

$$= \underline{\underline{u^2v}}$$

(16) $y_1 = 1 - x_1$; $y_2 = x_1(1 - x_2)$; $y_3 = x_1 x_2(1 - x_3)$ find $\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)}$

Sol: $y_1 = 1 - x_1$ $y_2 = x_1 - x_1 x_2$ $y_3 = x_1 x_2 - x_1 x_2 x_3$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$y_1, y_2, y_3 \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases}$

$$\begin{array}{l|l|l} y_1 = 1 - x_1 & y_2 = x_1 - x_1 x_2 & y_3 = x_1 x_2 - x_1 x_2 x_3 \\ \hline \frac{\partial y_1}{\partial x_1} = -1 & \frac{\partial y_2}{\partial x_1} = 1 - x_2 & \frac{\partial y_3}{\partial x_1} = x_2 - x_2 x_3 \\ \frac{\partial y_1}{\partial x_2} = 0 & \frac{\partial y_2}{\partial x_2} = 0 - x_1 & \frac{\partial y_3}{\partial x_2} = x_1 - x_1 x_3 \\ \frac{\partial y_1}{\partial x_3} = 0 & \frac{\partial y_2}{\partial x_3} = 0 & \frac{\partial y_3}{\partial x_3} = 0 - x_1 x_2 \end{array}$$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} -1 & 0 & 0 \\ 1 - x_2 & -x_1 & 0 \\ x_2 - x_2 x_3 & x_1 - x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= -1(x_1^2 x_2 - 0) - 0 + 0$$

$$= -x_1^2 x_2$$

(17) $u = x + y + z$; $u^2 v = y + z$; $u^3 w = z$ prove that $\frac{\partial(uvw)}{\partial(xyz)} = u^{-5}$

Sol: $u = x + y + z$ $u^2 v = y + z$ $u^3 w = z$
 $u = x + u^2 v$ $u^2 v = y + u^3 w$ $z = u^3 w$
 $x = u - u^2 v$ $y = u^2 v - u^3 w$

$xyz \begin{cases} u \\ v \\ w \end{cases}$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\begin{array}{l|l|l} x = u - u^2 v & y = u^2 v - u^3 w & z = u^3 w \\ \hline \frac{\partial x}{\partial u} = 1 - 2uv & \frac{\partial y}{\partial u} = 2uv - 3u^2 w & \frac{\partial z}{\partial u} = 3u^2 w \end{array}$$

$$\frac{dx}{dv} = 0 - v^2 \quad \left| \quad \frac{dy}{dv} = v^2 - 0 \quad \right| \quad \frac{dz}{dv} = 0$$

$$\frac{dx}{dw} = 0 \quad \left| \quad \frac{dy}{dw} = 0 - v^3 \quad \right| \quad \frac{dz}{dw} = v^3$$

$$\frac{d(xyz)}{d(uvw)} = \begin{vmatrix} 1-2uv & -v^2 & 0 \\ 2uv-3v^2w & v^2 & -v^3 \\ 3v^2w & 0 & v^3 \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2uv-3v^2w & v^2 & -v^3 \\ 3v^2w & 0 & v^3 \end{vmatrix}$$

$$= v^3 \begin{vmatrix} 1 & 0 & 0 \\ 2uv-3v^2w & v^2 & -1 \\ 3v^2w & 0 & 1 \end{vmatrix}$$

$$= v^3 [1(v^2+0) - 0+0]$$

$$= v^3 (v^2)$$

$$\frac{d(xyz)}{d(uvw)} = v^5$$

We know that, $\frac{d(uvw)}{d(xyz)} \cdot \frac{d(xyz)}{d(uvw)} = 1$

$$\frac{d(uvw)}{d(xyz)} \cdot v^5 = 1$$

$$\frac{d(uvw)}{d(xyz)} = \frac{1}{v^5}$$

$$\boxed{\frac{d(uvw)}{d(xyz)} = v^{-5}}$$

(18) If $u^3+v^3=x+y$; $u^2+v^2=x^3+y^3$ prove that $\frac{d(uv)}{d(xy)}$

Sol

Let us take $f_1 = u^3+v^3-x-y$

$$f_2 = u^2+v^2-x^3-y^3$$

$$f_1 = u^3+v^3-x-y$$

$$f_2 = u^2+v^2-x^3-y^3$$

$$\frac{df_1}{du} = 3u^2$$

$$\frac{df_1}{dv} = 3v^2$$

$$\frac{df_2}{du} = 2u$$

$$\frac{df_2}{dv} = 2v$$

$$\frac{\partial f_1}{\partial x} = -1$$

$$\frac{\partial f_1}{\partial y} = -1$$

$$\frac{\partial f_2}{\partial x} = -3x^2$$

$$\frac{\partial f_2}{\partial y} = -3y^2$$

We know that $\frac{d(f_1, f_2)}{d(xy)} = (-1)^2 \frac{\frac{d(f_1, f_2)}{d(xy)}}{\frac{d(f_1, f_2)}{d(uv)}}$

$$\frac{d(f_1, f_2)}{d(xy)} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= -3y^2 - 3x^2$$

$$\frac{d(f_1, f_2)}{d(uv)} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= 6u^2v - 6uv^2$$

$$\frac{d(uv)}{d(xy)} = \frac{-3y^2 - 3x^2}{6u^2v - 6uv^2} = \frac{1}{2} \frac{(y^2 - x^2)}{(u^2v - uv^2)}$$

Q. If $u = x(1-y)$, $v = xy$ prove that $\frac{d(uv)}{d(xy)} \times \frac{d(xy)}{d(uv)} = 1$.

$$u = x(1-y) \quad v = xy$$

$$J = \frac{d(uv)}{d(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{d(uv)}{d(xy)} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$$

$$u = x(1-y)$$

$$v = xy$$

$$\frac{\partial u}{\partial x} = 1-y$$

$$\frac{\partial v}{\partial x} = y$$

$$\frac{\partial u}{\partial y} = -x$$

$$\frac{\partial v}{\partial y} = x$$

$$= (1-y)x + xy$$

$$= x - xy + xy$$

$$= x$$

$$u = x - xy$$

$$v = xy$$

$$u = x - v$$

$$y = \frac{v}{x}$$

$$x = u + v$$

$$y = \frac{v}{u+v}$$

$$J' = \frac{d(xy)}{d(uv)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = u + v$$

$$y = \frac{v}{u+v}$$

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial y}{\partial u} = v \frac{-1}{(u+v)^2} = \frac{-v}{(u+v)^2}$$

$$\frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial v} = \frac{(u+v)(1) - v(0+1)}{(u+v)^2} = \frac{u}{(u+v)^2}$$

$$\frac{d(xy)}{d(uv)} = \begin{vmatrix} 1 & 1 \\ \frac{-v}{(u+v)^2} & \frac{u}{(u+v)^2} \end{vmatrix}$$

$$= \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x-xy+xy} = \frac{1}{x}$$

$$J \cdot J' = x \cdot \frac{1}{x} = 1$$

$$\therefore \frac{d(xy)}{d(xy)} \cdot \frac{d(xy)}{d(uv)} = 1$$

⑦ If $x = r \cos \theta$, $y = r \sin \theta$. Show that $\frac{d(xy)}{d(r\theta)} \cdot \frac{d(r\theta)}{d(xy)} = 1$

Sol:-

$$x = r \cos \theta \quad y = r \sin \theta$$

$$xy < r$$

$$J = \frac{d(xy)}{d(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$x = r \cos \theta$$

$$\frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial x}{\partial \theta} = r(-\sin \theta)$$

$$y = r \sin \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\frac{d(xy)}{d(r\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta$$

$$= r (\cos^2 \theta + \sin^2 \theta)$$

$$= r(1) = \underline{r}$$

$$x = r \cos \theta$$

S.O.B.

$$x^2 = r^2 \cos^2 \theta$$

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$$y = r \sin \theta$$

S.O.B.

$$y^2 = r^2 \sin^2 \theta$$

$$y/x = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

$$\theta = \tan^{-1}(y/x)$$

$$r\theta < xy$$

$$J' = \frac{d(r\theta)}{d(xy)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}}$$

$$= \frac{x}{x^2 + y^2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\sqrt{x^2 + y^2}} \left(\frac{\partial y}{\partial y} \right)$$

$$= \frac{y}{x^2 + y^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot y \left(\frac{-1}{x^2} \right)$$

$$= \frac{-y}{x^2 + y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2 + y^2}$$

$$J' = \frac{d(r\theta)}{d(xy)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{vmatrix}$$

$$= \frac{x^2}{\sqrt{x^2+y^2} (x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2} (x^2+y^2)}$$

$$= \frac{x^2+y^2}{\sqrt{x^2+y^2} (x^2+y^2)}$$

$$= \frac{1}{x}$$

$$\therefore J \cdot J' = x \cdot \frac{1}{x} = 1$$

$$\therefore \frac{d(xy)}{d(r\theta)} \cdot \frac{d(r\theta)}{d(xy)} = 1$$

⑤ If $x=uv$; $y=\frac{u}{v}$ prove that $\frac{d(xy)}{d(uv)} \times \frac{d(uv)}{d(xy)} = 1$.

Sol

$$x=uv \quad y=\frac{u}{v}$$

$$J = \frac{d(xy)}{d(uv)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x=uv$$

$$\frac{\partial x}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u$$

$$y=\frac{u}{v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{v}$$

$$\frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{d(xy)}{d(uv)} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= v \cdot \left(-\frac{u}{v^2}\right) - \frac{u}{v}$$

$$J = \frac{-2u}{v}$$

a $x=uv$ $y=\frac{u}{v}$

$$u = \frac{x}{v}$$

$$v = \frac{x}{uy}$$

$$u = \frac{xy}{v}$$

$$u^2 = xy \Rightarrow u = \sqrt{xy}$$

$$v = \frac{u}{y} \Rightarrow v = \frac{\sqrt{xy}}{y}$$

$$v = \frac{\sqrt{x}}{\sqrt{y}}$$

$$v^2 = \frac{x}{y} \Rightarrow v = \frac{\sqrt{x}}{\sqrt{y}}$$

$$J' = \frac{d(uv)}{d(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$u = \sqrt{xy}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{xy}} y$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{xy}} x$$

$$v = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{y}} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{\partial v}{\partial y} = -\frac{\sqrt{x}}{2y\sqrt{y}}$$

$$\frac{d(uv)}{d(xy)} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$

$$= \frac{-y\sqrt{x}}{4y\sqrt{xy}} - \frac{x}{4(xy)^{3/2}}$$

$$= \frac{-1}{4y} - \frac{x}{4xy^2}$$

$$= \frac{-1}{4y} - \frac{1}{4y} = \frac{-2}{4y} = \frac{-1}{2y}$$

$$J \cdot J' = \frac{-2y}{4y} \times \frac{-1}{2y} = \frac{-1}{2y} \times \frac{-1}{2y} = \frac{1}{4y^2}$$

$$= 1$$

$$\therefore \frac{d(xy)}{d(uv)} \times \frac{d(uv)}{d(xy)} = 1$$

⑥ If $x = r \cos \theta$, $y = r \sin \theta$ show that $\frac{d(xy)}{d(r\theta)} = r$.

Sol:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$xy < r$$

$$J = \frac{d(xy)}{d(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{d(xy)}{d(r\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta$$

$$= r \cos^2 \theta + r \sin^2 \theta$$

$$= r (\cos^2 \theta + \sin^2 \theta)$$

$$\boxed{\frac{d(xy)}{d(r\theta)} = r}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 = r^2 \cos^2 \theta$$

$$y^2 = r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$x^2 + y^2 = r^2$$

$$r = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta}$$

$$\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$J = \frac{d(r\theta)}{d(xy)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$r = \sqrt{x^2 + y^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} (2x)$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}} (2y)$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot y \left(\frac{-1}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{x^2+y^2\sqrt{x^2+y^2}} + \frac{y^2}{x^2+y^2\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2}{x^2+y^2\sqrt{x^2+y^2}} = \frac{1}{\sqrt{r}}$$

⑧ If $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ evaluate $\frac{\partial(xyz)}{\partial(r\theta z)}$

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$xyz < \frac{r_0}{z}$$

$$\frac{\partial(xyz)}{\partial(r\theta z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\begin{array}{l} x = r \cos \theta \\ \frac{\partial x}{\partial r} = \cos \theta \\ \frac{\partial x}{\partial \theta} = r(-\sin \theta) \\ \frac{\partial x}{\partial z} = 0 \end{array} \quad \begin{array}{l} y = r \sin \theta \\ \frac{\partial y}{\partial r} = \sin \theta \\ \frac{\partial y}{\partial \theta} = r \cos \theta \\ \frac{\partial y}{\partial z} = 0 \end{array} \quad \begin{array}{l} z = z \\ \frac{\partial z}{\partial r} = 0 \\ \frac{\partial z}{\partial \theta} = 0 \\ \frac{\partial z}{\partial z} = 1 \end{array}$$

$$\frac{\partial(xyz)}{\partial(r\theta z)} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos \theta [r \cos \theta - 0] + r \sin \theta [\sin \theta - 0]$$

$$= r \cos^2 \theta + r \sin^2 \theta$$

$$= r (\cos^2 \theta + \sin^2 \theta)$$

$$= \underline{\underline{r}}$$

⑨ If $u = 2xy$, $v = x^2 - y^2$, $x = r \cos \theta$, $y = r \sin \theta$ evaluate

$$\frac{\partial(uv)}{\partial(r\theta)}$$

Sol

$$u = 2xy, \quad v = x^2 - y^2, \quad x = r \cos \theta, \quad y = r \sin \theta$$

$$uv < \frac{x}{y} > r_0$$

$$\frac{\partial(uv)}{\partial(r\theta)} = \frac{\partial(uv)}{\partial(xy)} \cdot \frac{\partial(xy)}{\partial(r\theta)}$$

$$\frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$u = 2xy \quad v = x^2 - y^2 \quad = -4y^2 - 4x^2$$

$$\frac{\partial u}{\partial x} = 2y \quad \frac{\partial v}{\partial x} = 2x \quad = -4(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = 2x \quad \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$x = r\cos\theta \quad y = r\sin\theta \quad = r\cos^2\theta + r\sin^2\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial y}{\partial r} = \sin\theta \quad = r(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta \quad = r$$

$$= \sqrt{x^2 + y^2}$$

$$\therefore \frac{\partial(uv)}{\partial(r\theta)} = -4(x^2 + y^2) \cdot \sqrt{x^2 + y^2} = -4(x^2 + y^2)^{3/2}$$

$$= -4(r^2)^{3/2} = -4r^3$$

⑩ If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\cos\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$. Then evaluate $\frac{\partial(xyz)}{\partial(r\theta\phi)}$

$$\frac{\partial(xyz)}{\partial(r\theta\phi)} = \frac{\partial(xyz)}{\partial(uvw)} \cdot \frac{\partial(uvw)}{\partial(r\theta\phi)}$$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x, y, z \rightarrow uvw \rightarrow r\theta\phi$

$$x = \sqrt{vw} \quad y = \sqrt{wu} \quad z = \sqrt{uv}$$

$$\frac{\partial x}{\partial u} = 0 \quad \frac{\partial y}{\partial u} = \frac{1}{2\sqrt{wu}} \quad \frac{\partial z}{\partial u} = \frac{1}{2\sqrt{uv}}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2\sqrt{vw}} \quad \frac{\partial y}{\partial v} = 0 \quad \frac{\partial z}{\partial v} = \frac{1}{2\sqrt{uv}}$$

$$\frac{\partial x}{\partial w} = \frac{1}{2\sqrt{vw}} \quad \frac{\partial y}{\partial w} = \frac{1}{2\sqrt{wu}} \quad \frac{\partial z}{\partial w} = 0$$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} 0 & \frac{1}{2\sqrt{vw}} & \frac{1}{2\sqrt{vw}} \\ \frac{1}{2\sqrt{wu}} & 0 & \frac{1}{2\sqrt{wu}} \\ \frac{1}{2\sqrt{uv}} & \frac{1}{2\sqrt{uv}} & 0 \end{vmatrix}$$

$$= \frac{1}{2\sqrt{vw}} \cdot \frac{1}{2\sqrt{wu}} \cdot \frac{1}{2\sqrt{uv}} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{8uvw} [-1(0-1) + 1(1-0)]$$

$$= \frac{1}{8uvw} (1+1)$$

$$= \frac{1}{4uvw} (2) = \frac{1}{2uvw}$$

$$\frac{\partial(uvw)}{\partial(r\theta\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix}$$

$$\begin{aligned} u &= r \sin \theta \cos \phi & v &= r \sin \theta \sin \phi & w &= r \cos \theta \\ \frac{\partial u}{\partial r} &= \sin \theta \cos \phi & \frac{\partial v}{\partial r} &= \sin \theta \sin \phi & \frac{\partial w}{\partial r} &= \cos \theta \\ \frac{\partial u}{\partial \theta} &= r \cos \theta \cos \phi & \frac{\partial v}{\partial \theta} &= r \sin \theta \cos \phi & \frac{\partial w}{\partial \theta} &= r(-\sin \theta) \\ \frac{\partial u}{\partial \phi} &= r \sin \theta (-\sin \phi) & \frac{\partial v}{\partial \phi} &= r \sin \theta \cos \phi & \frac{\partial w}{\partial \phi} &= 0 \end{aligned}$$

$$\frac{\partial(uvw)}{\partial(r\theta\phi)} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & -r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= r \cdot r \begin{vmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \sin \theta \sin \phi & -\cos \theta \sin \phi & \sin \theta \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{vmatrix}$$

$$= r^2 [\sin \theta \cos \phi (0 + \sin \theta \cos \phi) - \cos \theta \cos \phi (0 - \sin \theta \cos \theta \cos \phi) - \sin \theta \sin \phi (-\sin \theta \sin \phi + \cos^2 \theta \sin \phi)]$$

$$= r^2 [\sin^2 \theta \cos^2 \phi + \sin \theta \cos^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi - \sin \theta \cos^2 \theta \sin \phi]$$

$$= r^2 [\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \sin \theta \cos^2 \theta (\cos^2 \phi - \sin^2 \phi)]$$

$$= r^2 [\sin^2 \theta + \sin \theta \cos^2 \theta]$$

⑪ $y_1 = \frac{x_2 x_3}{x_1}$; $y_2 = \frac{x_3 x_1}{x_2}$; $y_3 = \frac{x_1 x_2}{x_3}$ show that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$.

Sol:- $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$ $y_1, y_2, y_3 \begin{matrix} \swarrow \\ \nwarrow \end{matrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$

$$y_1 = \frac{x_2 x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_1} = x_2 x_3 \left(-\frac{1}{x_1^2} \right)$$

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$y_2 = \frac{x_3 x_1}{x_2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_2} = x_3 x_1 \left(-\frac{1}{x_2^2} \right)$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$y_3 = \frac{x_1 x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$$

$$\frac{\partial y_3}{\partial x_3} = x_1 x_2 \left(-\frac{1}{x_3^2} \right)$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_1 x_3}{x_1^2} & \frac{x_1 x_2}{x_1^2} \\ \frac{x_2 x_3}{x_2^2} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1 x_2}{x_2^2} \\ \frac{x_2 x_3}{x_3^2} & \frac{x_1 x_3}{x_3^2} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2} \cdot \frac{1}{x_2^2} \cdot \frac{1}{x_3^2} (x_1 x_2)(x_1 x_3)(x_2 x_3) \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} [-1(1-1) - 1(-1-1) + 1(1+1)]$$

$$= -1(0) - 1(-2) + 1(2)$$

$$= 0 + 2 + 2$$

$$\frac{d(y_1 y_2 y_3)}{d(x_1 x_2 x_3)} = \underline{\underline{4}}$$

⑬ $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$ find $\frac{d(uv)}{d(xy)}$

$$\frac{d(uv)}{d(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad uv < \begin{matrix} x \\ y \end{matrix}$$

$$u = \frac{y^2}{2x} \quad \left| \quad v = \frac{x^2+y^2}{2x} \right.$$

$$\frac{\partial u}{\partial x} = \frac{y^2}{2} \left(\frac{-1}{x^2} \right) \quad \frac{\partial v}{\partial x} = \frac{2x(2x+0) - (x^2+y^2)2}{(2x)^2} = \frac{4x^2 - 2x^2 - 2y^2}{4x^2} = \frac{2(x^2 - y^2)}{4x^2} = \frac{x^2 - y^2}{2x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2x} (2y) = \frac{y}{x} \quad \frac{\partial v}{\partial y} = \frac{1}{2x} (2y) = \frac{y}{x}$$

$$\frac{d(uv)}{d(xy)} = \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2-y^2}{2x^2} & \frac{y}{x} \end{vmatrix}$$

$$= \frac{1}{2x^2} \cdot \frac{y}{x} \begin{vmatrix} -y^2 & 1 \\ x^2-y^2 & 1 \end{vmatrix}$$

$$= \frac{y}{2x^3} [-y^2 - x^2 + y^2]$$

$$= \frac{-x^2 y}{2x^3} = \underline{\underline{-\frac{y}{2x}}}$$

⑮ $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$ show that

$$\frac{d(uvw)}{d(xy z)} = (x-y)(y-z)(z-x)$$

$$\frac{d(uvw)}{d(xy z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$uvw < \begin{matrix} x \\ y \\ z \end{matrix}$$

$$\begin{array}{l|l|l}
 u = xyz & v = xy + yz + zx & w = x + y + z \\
 \frac{\partial u}{\partial x} = yz & \frac{\partial v}{\partial x} = y + 0 + z = y + z & \frac{\partial w}{\partial x} = 1 + 0 + 0 = 1 \\
 \frac{\partial u}{\partial y} = xz & \frac{\partial v}{\partial y} = x + z + 0 = x + z & \frac{\partial w}{\partial y} = 0 + 1 + 0 = 1 \\
 \frac{\partial u}{\partial z} = xy & \frac{\partial v}{\partial z} = 0 + y + x = x + y & \frac{\partial w}{\partial z} = 0 + 0 + 1 = 1
 \end{array}$$

$$\begin{aligned}
 \frac{d(uvw)}{d(xyz)} &= \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix} \\
 &= yz(z+x-x-y) - zx(y+z-x-y) + xy(y+z-x-y) \\
 &= yz(z-y) - zx(z-x) + xy(y-x) \\
 &= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y \\
 &= x^2y^2
 \end{aligned}$$

(19) If $x^2 + y^2 + u^2 + v^2 = 0$ and $uv + xy = 0$, prove that $\frac{d(uv)}{d(xy)} = \frac{x^2 - y^2}{u^2 - v^2}$.
 Let us take $f_1 = x^2 + y^2 + u^2 + v^2$, $f_2 = uv + xy$.

$$\begin{array}{l|l}
 \frac{\partial f_1}{\partial x} = 2x & \frac{\partial f_2}{\partial x} = y \\
 \frac{\partial f_1}{\partial y} = 2y & \frac{\partial f_2}{\partial y} = x \\
 \frac{\partial f_1}{\partial u} = 2u & \frac{\partial f_2}{\partial u} = v \\
 \frac{\partial f_1}{\partial v} = 2v & \frac{\partial f_2}{\partial v} = u
 \end{array}$$

We know that,

$$\frac{d(uv)}{d(xy)} = (-1)^2 \frac{\frac{d(f_1 f_2)}{d(xy)}}{\frac{d(f_1 f_2)}{d(uv)}}$$

$$\begin{aligned}
 \frac{d(f_1 f_2)}{d(xy)} &= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} \\
 &= 2x^2 - 2y^2 \\
 &= 2(x^2 - y^2)
 \end{aligned}$$

$$\begin{aligned}
 \frac{d(f_1 f_2)}{d(uv)} &= \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} \\
 &= 2u^2 + 2v^2 \\
 &= 2(u^2 + v^2)
 \end{aligned}$$

$$\therefore \frac{d(uv)}{d(xy)} = (-1)^2 \frac{d(x^2-y^2)}{d(u^2+v^2)} = \frac{x^2-y^2}{u^2+v^2}$$

3/12/2019
Tuesday

Functional Dependence

② If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$.

$$J = \frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \frac{u}{v} > xy$$

$$u = \frac{x+y}{1-xy}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{(1-xy)(1) - (x+y)(0-y)}{(1-xy)^2} \\ &= \frac{1-xy + xy + y^2}{(1-xy)^2} \\ &= \frac{1+y^2}{(1-xy)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{(1-xy)(1) - (x+y)(1-x)}{(1-xy)^2} \\ &= \frac{1-xy + x^2 + xy}{(1-xy)^2} \\ &= \frac{1+x^2}{(1-xy)^2} \end{aligned}$$

$$\frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} [1-1]$$

$$= \frac{1}{(1-xy)^2} (0)$$

$$\boxed{\frac{\partial(uv)}{\partial(xy)} = 0}$$

$$v = \tan^{-1}x + \tan^{-1}y$$

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{1}{1+x^2} + 0 \\ &= \frac{1}{1+x^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial y} &= 0 + \frac{1}{1+y^2} \\ &= \frac{1}{1+y^2} \end{aligned}$$

$\therefore U$ and V are functionally dependent.

That is, there is a relation b/w U and V .

$$V = \tan^{-1}x + \tan^{-1}y$$

$$\tan V = \tan(\tan^{-1}x + \tan^{-1}y)$$

$$= \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x) \cdot \tan(\tan^{-1}y)}$$

$$= \frac{x+y}{1-xy}$$

$$\tan V = U$$

② If $U = x+y+z$, $U^2V = y+z$, $U^3\omega = z$.

$$U = x+y+z$$

$$U^2V = y+z$$

$$U^3\omega = z$$

$$U = x+U^2V$$

$$U^2V = y+U^3\omega$$

$$z = U^3\omega$$

$$x = U - U^2V$$

$$y = U^2V - U^3\omega$$

$$J = \frac{d(x,y,z)}{d(U,V,\omega)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial U} & \frac{\partial z}{\partial V} & \frac{\partial z}{\partial \omega} \end{vmatrix}$$

$$\begin{matrix} x \\ y \\ z \end{matrix} \rightarrow U, V, \omega$$

$$x = U - U^2V$$

$$\frac{\partial x}{\partial U} = 1 - V(2U) = 1 - 2UV$$

$$\frac{\partial x}{\partial V} = 0 - U^2(1) = -U^2$$

$$\frac{\partial x}{\partial \omega} = 0$$

$$y = U^2V - U^3\omega$$

$$\frac{\partial y}{\partial U} = V(2U) - \omega(3U^2) = 2UV - 3U^2\omega$$

$$\frac{\partial y}{\partial V} = U^2(1) - 0 = U^2$$

$$\frac{\partial y}{\partial \omega} = 0 - U^3 = -U^3$$

$$z = U^3\omega$$

$$\frac{\partial z}{\partial U} = \omega(3U^2) = 3U^2\omega$$

$$\frac{\partial z}{\partial V} = 0$$

$$\frac{\partial z}{\partial \omega} = U^3(1) = U^3$$

$$J = \frac{d(x,y,z)}{d(U,V,\omega)} = \begin{vmatrix} 1-2UV & -U^2 & 0 \\ 2UV-3U^2\omega & U^2 & -U^3 \\ 3U^2\omega & 0 & U^3 \end{vmatrix}$$

$$= \cancel{1-2UV} U^2 \cdot U^3 \begin{vmatrix} 1-2UV & -1 & 0 \\ 2UV-3U^2\omega & 1 & -1 \\ 3U^2\omega & 0 & 1 \end{vmatrix}$$

$$= U^5 [1 - 2uv(1+0) + 1(2uv - 3v^2w + 3u^2w) + 0]$$

$$= U^5 [1 - 2uv + 2uv - 3v^2w + 3u^2w]$$

$$\frac{d(xy, z)}{d(u, v, w)} = U^5. \quad \text{Hence}$$

$\therefore xy, z$ are not functionally dependent.

Hence there is no relation between x, y and z .

④ If $U = \frac{x-y}{x+y}, V = \frac{xy}{(x+y)^2}$

sd:- $U = \frac{x-y}{x+y}, V = \frac{xy}{(x+y)^2}$

$$J = \frac{d(U, V)}{d(x, y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

~~U > xy~~

$$\frac{U}{V} > xy$$

$$U = \frac{x-y}{x+y}$$

$$\frac{\partial U}{\partial x} = \frac{(x+y)(1-0) - (x-y)(1+0)}{(x+y)^2} = \frac{x+y-x+y}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(x+y)(0-1) - (x-y)(0+1)}{(x+y)^2} = \frac{-x-y-x+y}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$\frac{d(U, V)}{d(x, y)} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{y(y^2-x^2)}{(x+y)^4} & \frac{x(x^2-y^2)}{(x+y)^4} \end{vmatrix}$$

$$= \frac{2xy}{(x+y)^2} \begin{vmatrix} 1 & -1 \\ \frac{y^2-x^2}{(x+y)^2} & \frac{x^2-y^2}{(x+y)^2} \end{vmatrix}$$

$$= \frac{2xy}{(x+y)^2} \left[\frac{x^2-y^2}{(x+y)^2} + \frac{y^2-x^2}{(x+y)^2} \right]$$

$$= \frac{2xy}{(x+y)^2} \left[\frac{x^2-y^2+y^2-x^2}{(x+y)^2} \right]$$

$$= \frac{2xy}{(x+y)^2} (0)$$

$$V = \frac{xy}{(x+y)^2}$$

$$\frac{\partial V}{\partial x} = \frac{(x+y)^2 y - xy \cdot 2(x+y)}{[(x+y)^2]^2}$$

$$= \frac{x^2y + y^3 + 2xy^2 - 2x^2y - 2xy^2}{(x+y)^4}$$

$$= \frac{y^3 - x^2y}{(x+y)^4}$$

$$\frac{\partial V}{\partial y} = \frac{(x+y)^2 x - xy \cdot 2(x+y)}{[(x+y)^2]^2}$$

$$= \frac{(x^2+y^2+2xy)x - 2x^2y - 2xy^2}{(x+y)^4}$$

$$= \frac{x^3 + xy^2 + 2x^2y - 2x^2y - 2xy^2}{(x+y)^4}$$

$$= \frac{x^3 - xy^2}{(x+y)^4}$$

$$\therefore \frac{d(UV)}{d(xy)} = 0$$

⑤ $U = xy + yz + zx$, $V = x^2 + y^2 + z^2$, $W = x + y + z$.

$$J = \frac{d(UVW)}{d(xy z)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix} \quad U, V, W \begin{matrix} \swarrow \\ \downarrow \\ \searrow \end{matrix} \begin{matrix} x \\ y \\ z \end{matrix}$$

$$\begin{array}{l|l|l} U = xy + yz + zx & V = x^2 + y^2 + z^2 & W = x + y + z \\ \hline \frac{\partial U}{\partial x} = y + 0 + z & \frac{\partial V}{\partial x} = 2x & \frac{\partial W}{\partial x} = 1 \\ \frac{\partial U}{\partial y} = x + z & \frac{\partial V}{\partial y} = 2y & \frac{\partial W}{\partial y} = 1 \\ \frac{\partial U}{\partial z} = y + x & \frac{\partial V}{\partial z} = 2z & \frac{\partial W}{\partial z} = 1 \end{array}$$

$$\frac{d(UVW)}{d(xy z)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$\begin{aligned} &= y+z(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y) \\ &= 2y^2 - 2yz + 2yz - 2z^2 - 2x^2 + 2zx - 2zx + 2z^2 + 2xy - 2y^2 + 2xy \\ &= 0. \end{aligned}$$

$$\therefore \frac{d(UVW)}{d(xy z)} = 0$$

① If $U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $V = \sin^{-1}x + \sin^{-1}y$. Show that U, V are functionally dependent.

Sol:-

$$J = \frac{d(UV)}{d(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} \quad U, V < \frac{\pi}{2}$$

$$U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\frac{\partial U}{\partial x} = \sqrt{1-y^2} + y \cdot \frac{1}{2\sqrt{1-x^2}} (-2x)$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial U}{\partial y} = x \cdot \frac{1}{2\sqrt{1-y^2}} (-2y) + \sqrt{1-x^2}$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$V = \sin^{-1}x + \sin^{-1}y$$

$$\frac{\partial V}{\partial x} = \frac{1}{\sqrt{1-x^2}} + 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\partial V}{\partial y} = 0 + \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\frac{d(UV)}{d(xy)} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}} \left[\sqrt{1-x^2} \cdot \sqrt{1-y^2} - xy - \left(\frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \right) \sqrt{1-x^2} \right]$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \left[\sqrt{1-x^2} \cdot \sqrt{1-y^2} - xy + xy - \sqrt{1-x^2} \cdot \sqrt{1-x^2} \right]$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \quad (0)$$

$$= 0$$

$$\therefore \frac{\partial(UV)}{\partial(xy)} = 0$$

$\therefore U, V$ are functionally dependent.

i.e., there is a relation b/w U and V .

$$U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$x = \sin y \Rightarrow y = \sin^{-1} x$$

$$y = \sin x \Rightarrow x = \sin^{-1} y$$

$$= \sin y \sqrt{1 - \sin^2 x} + \sin x \sqrt{1 - \sin^2 y}$$

$$= \sin y \cdot \cos x + \sin x \cdot \cos y$$

$$= \sin(x+y)$$

$$= \sin(\sin^{-1} y + \sin^{-1} x)$$

$$\boxed{U = \sin y}$$

Maxima And Minima (without constraints)

$$\textcircled{2} \quad x^3 y^2 (1-x-y)$$

Sol:-

$$\text{Let } f(x, y) = x^3 y^2 (1-x-y)$$

$$f(x, y) = x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$\frac{\partial f}{\partial x} = y^2(3x^2) - y^2(4x^3) - y^3(3x^2)$$

$$= 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$\frac{\partial f}{\partial y} = x^3(2y) - x^4(2y) - x^3(3y^2)$$

$$= 2x^3 y - 2x^4 y - 3x^3 y^2$$

$$\text{we have } \frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$$

$$x^2 y^2 (3 - 4x - 3y) = 0$$

$$x=0, y=0, 4x+3y-3=0$$

$$\text{if } x=0, 2x+3y-2=0$$

$$3y-2=0$$

$$\boxed{y = \frac{2}{3}}$$

$$(0, \frac{2}{3})$$

$$2x^3 y - 2x^4 y - 3x^3 y^2 = 0$$

$$x^3 y (2 - 2x - 3y) = 0$$

$$x=0, y=0, (2x+3y-2)=0$$

$$\text{if } y=0, 2x+3y-2=0$$

$$2x-2=0$$

$$\boxed{x=1}$$

$$(1, 0)$$

$$\text{If } 4x+3y-3=0, x=0$$

$$3y-3=0$$

$$\boxed{y=1}$$

$$(0,1)$$

$$\text{If } 4x+3y-3=0, y=0$$

$$4x-3=0$$

$$\boxed{x=3/4}$$

$$(3/4, 0)$$

$$\text{If } 4x+3y-3=0, 2x+3y-2=0$$

$$4x+3y-3=0$$

$$-2x+3y-2=0$$

$$2x-1=0$$

$$\boxed{x=1/2}$$

$$4(1/2)+3y-3=0$$

$$2+3y-3=0$$

$$\boxed{y=1/3}$$

$$(1/2, 1/3)$$

$\therefore$ The stationary points are $(0, 2/3), (1, 0), (0, 1), (3/4, 0), (1/2, 1/3)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y - 6xy^3$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

At the point $(0, 2/3)$

$$r=0, s=0, t=0, rt-s^2=0$$

At the point $(1, 0)$

$$r=0, s=0, t=0, rt-s^2=0$$

At the point $(0, 1)$

$$r=0, s=0, t=0, rt-s^2=0$$

At the point $(3/4, 0)$

$$r=0, s=0, t=2(3/4)^3 - 2(3/4)^4, rt-s^2=0$$

$$= \frac{27}{128}$$

At the point $(1/2, 1/3)$

$$r = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3) - 6(1/2)(1/3)^3$$

$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9}$$

$$s = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^2$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{12}$$

$$t = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$\Rightarrow r_t - s^v = \left(-\frac{1}{4}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2$$

$$= \frac{1}{32} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$$

$$r_t - s^v > 0, \quad r = -\frac{1}{4} < 0.$$

\(\therefore\) The function has maximum at the point \(\left(\frac{1}{2}, \frac{1}{3}\right)\).

Maximum value is $f = x^3 y^2 (1-x-y)$

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{72} \left(\frac{6-3-2}{6}\right)$$

$$= \frac{1}{72} \left(\frac{1}{6}\right) = \underline{\underline{\frac{1}{432}}}$$

④ $\sin x + \sin y + \sin(x+y)$

Let $f(x, y) = \sin x + \sin y + \sin(x+y)$

$$\frac{df}{dx} = \cos x + \cos(x+y)$$

$$\frac{df}{dy} = \cos y + \cos(x+y)$$

We have $\frac{df}{dx} = 0, \quad \frac{df}{dy} = 0$

$$\cos x + \cos(x+y) = 0$$

$$2 \cos\left(\frac{x+x+y}{2}\right) \cos\left(\frac{x-x-y}{2}\right) = 0$$

$$\cos\left(\frac{2x+y}{2}\right) \cdot \cos\left(-\frac{y}{2}\right) = 0$$

$$\cos \frac{2x+y}{2} = 0, \quad \cos y/2 = 0$$

$$\frac{2x+y}{2} = \cos^{-1}(0) \quad y/2 = \cos^{-1}(0)$$

$$\frac{2x+y}{2} = \frac{\pi}{2}, \frac{3\pi}{2} \dots \quad y/2 = \frac{\pi}{2}, \frac{3\pi}{2} \dots$$

$$2x+y = \pi, 3\pi \dots \quad y = \pi, 3\pi \dots$$

$$2x+y = \pi, \quad 2x+y = 3\pi, \quad y = \pi, \quad y = 3\pi$$

$$x+y = \pi, \quad x+y = 3\pi, \quad x = \pi, \quad x = 3\pi$$

$$\cos y + \cos(x+y) = 0$$

$$2 \cos\left(\frac{y+x+y}{2}\right) \cos\left(\frac{y-x-y}{2}\right) = 0$$

$$\cos\left(\frac{x+2y}{2}\right) \cdot \cos\left(-\frac{x}{2}\right) = 0$$

$$\cos\left(\frac{x+2y}{2}\right) = 0 \quad \cos x/2 = 0$$

$$\frac{x+2y}{2} = \cos^{-1}(0) \quad x/2 = \cos^{-1}(0)$$

$$\frac{x+2y}{2} = \frac{\pi}{2}, \frac{3\pi}{2} \dots \quad x/2 = \frac{\pi}{2}, \frac{3\pi}{2} \dots$$

$$x+2y = \pi, 3\pi \dots \quad x = \pi, 3\pi \dots$$

$$\text{if } 2x+y=\pi, \quad x+2y=3\pi$$

$$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$\begin{aligned} 2x+y &= \pi \\ 2x+4y &= 3\pi \\ \hline -3y &= -2\pi \\ y &= \frac{2\pi}{3} \end{aligned}$$

$$\begin{aligned} 2x + \frac{\pi}{3} &= \pi \\ 2x &= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \\ x &= \frac{\pi}{3} \end{aligned}$$

$$\text{if } 2x+y=\pi, \quad x+2y=3\pi$$

$$\left(-\frac{\pi}{3}, \frac{5\pi}{3}\right)$$

$$\begin{aligned} \text{if } 2x+y &= 3\pi, \quad x=3\pi \\ 6\pi+y &= 3\pi \Rightarrow y = -3\pi \\ (3\pi, -3\pi) \end{aligned}$$

$$\text{if } 2x+y=\pi, \quad x=\pi$$

$$\begin{aligned} 2\pi+y &= \pi & 2\pi-\pi &= \pi \\ y &= -\pi & 2\pi &= 2\pi \\ x &= \pi \end{aligned}$$

$$\left(\pi, -\pi\right)$$

$$\text{if } 2x+y=\pi, \quad x=3\pi$$

$$6\pi+y=\pi \Rightarrow y = -5\pi$$

$$\begin{aligned} 2x-5\pi &= \pi \Rightarrow 2x = 6\pi \\ x &= 3\pi \end{aligned}$$

$$\left(3\pi, -5\pi\right)$$

$$\text{if } 2x+y=3\pi, \quad x+2y=\pi$$

$$\left(\frac{5\pi}{3}, -\frac{\pi}{3}\right)$$

$$\text{if } y=\pi, \quad x+2y=3\pi$$

$$x+2\pi=3\pi \Rightarrow x=\pi$$

$$\left(\pi, \pi\right)$$

$$\text{if } 2x+y=3\pi, \quad x+2y=3\pi$$

$$\left(\pi, \pi\right)$$

$$\text{if } 2x+y=3\pi, \quad x=\pi$$

$$\left(\pi, \pi\right)$$

$$\text{if } y=\pi, \quad x+2y=\pi$$

$$x+2\pi=\pi \Rightarrow x=-\pi$$

$$\left(-\pi, \pi\right)$$

$$\text{if } y=3\pi, \quad x+2y=\pi$$

$$x+6\pi=\pi \Rightarrow x=-5\pi$$

$$\left(-5\pi, 3\pi\right)$$

$$\text{if } y=3\pi, \quad x+2y=3\pi$$

$$x+6\pi=3\pi \Rightarrow x=-3\pi$$

$$\left(-3\pi, 3\pi\right)$$

$\therefore$ The stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), \left(-\frac{\pi}{3}, \frac{5\pi}{3}\right), \left(\pi, -\pi\right), \left(\frac{5\pi}{3}, -\frac{\pi}{3}\right), \left(\pi, \pi\right), \left(\pi, \pi\right), \left(-\pi, \pi\right), \left(\pi, \pi\right), \left(-5\pi, 3\pi\right), \left(-3\pi, 3\pi\right), \left(3\pi, -5\pi\right)$

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$\text{At } \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$r = -\sin \frac{\pi}{3} - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= -\frac{\sqrt{3}}{2} - \sin \frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - \sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$s = -\sin(\pi/3 + \pi/3) = -\sin 2\pi/3 = -\sin \pi/3 = -\frac{\sqrt{3}}{2}$$

$$t = -\sin \pi/3 - \sin(\pi/3 + \pi/3) \\ = -\frac{\sqrt{3}}{2} - \sin 2\pi/3 = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 \\ = 3 - \frac{3}{4} \\ = \frac{12-3}{4} = \frac{9}{4} > 0$$

$$\therefore rt - s^2 > 0, \quad r < 0, \quad \therefore \boxed{r = -\sqrt{3}}$$

$\therefore$ The function has maximum at point $(\pi/3, \pi/3)$.

$\therefore$ Maximum value $f = \sin x + \sin y + \sin(x+y)$

$$= \sin \pi/3 + \sin \pi/3 + \sin(\pi/3 + \pi/3) \\ = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin 2\pi/3 \\ = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin \pi/3 \\ = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\ = \frac{3\sqrt{3}}{2}$$

At $(-\pi/3, 5\pi/3)$

$$r = -\sin(-\pi/3) - \sin(-\pi/3 + 5\pi/3) \\ = \sin \pi/3 - \sin(4\pi/3) \\ = \frac{\sqrt{3}}{2} - \sin(\pi + \pi/3) \\ = \frac{\sqrt{3}}{2} + \sin \pi/3 \\ = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \underline{\underline{\sqrt{3}}}$$

$$s = -\sin(-\pi/3 + 5\pi/3) \\ = -\sin(4\pi/3) \\ = -\sin(\pi + \pi/3) \\ = \sin \pi/3 \\ = \underline{\underline{\frac{\sqrt{3}}{2}}}$$

$$t = -\sin(5\pi/3) - \sin(-\pi/3 + 5\pi/3) \\ = -\sin(2\pi - \pi/3) - \sin(4\pi/3) \\ = -\sin \pi/3 - \sin(\pi + \pi/3) \\ = -\sin \pi/3 + \sin \pi/3 = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \underline{\underline{\sqrt{3}}}$$

$$\begin{aligned}
 r^2 - s^2 &= (\sqrt{3})(\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 \\
 &= 3 - \frac{3}{4} = \frac{12-3}{4} = \frac{9}{4} > 0.
 \end{aligned}$$

$$\therefore r^2 - s^2 > 0, \quad r = \sqrt{3} > 0.$$

$\therefore$ The function has minimum at the point $(-\pi/3, 5\pi/3)$

$$\begin{aligned}
 \therefore \text{Minimum value } f &= \sin x + \sin y + \sin(x+y) \\
 &= \sin(-\pi/3) + \sin(5\pi/3) + \sin(-\pi/3 + 5\pi/3) \\
 &= -\sin \pi/3 + \sin(2\pi - \pi/3) + \sin(4\pi/3) \\
 &= -\frac{\sqrt{3}}{2} - \sin \pi/3 + \sin(\pi + \pi/3) \\
 &= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \sin \pi/3 \\
 &= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \\
 &= \underline{\underline{-\frac{3\sqrt{3}}{2}}}.
 \end{aligned}$$

At the points $(\pi, -\pi), (3\pi, -5\pi), (\pi, \pi), (\pi, \pi), (-\pi, \pi), (\pi, \pi), (-5\pi, 3\pi), (3\pi, 3\pi), (3\pi, -3\pi)$.

$$r^2 - s^2 = 0.$$

$\therefore$ We need further investigation.

$$\textcircled{4} \quad xy + \frac{a^3}{x} + \frac{a^3}{y}.$$

$$\text{Let } f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

$$\frac{df}{dx} = y + a^3 \left(\frac{1}{x^2}\right) + 0 = y - \frac{a^3}{x^2}$$

$$\frac{df}{dy} = x + 0 + a^3 \left(\frac{1}{y^2}\right) = x - \frac{a^3}{y^2}$$

$$\text{we have } \frac{df}{dx} = 0$$

$$\frac{df}{dy} = 0$$

$$y - \frac{a^3}{x^2} = 0$$

$$x - \frac{a^3}{y^2} = 0$$

$$y = \frac{a^3}{x^2}$$

$$x = \frac{a^3}{y^2}$$

Sub y value in $x = \frac{a^3}{y^2}$

$$x - \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} = 0$$

$$x - \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} x^4 = 0$$

$$x - \frac{x^4}{a^3} = 0$$

$$a^3x - x^4 = 0$$

$$x(a^3 - x^3) = 0$$

$$x = 0, (a^3 - x^3) = 0$$

$$x = 0, (a - x) = 0$$

$$\boxed{x = a}$$

Sub $x = a$ in $y = \frac{a^3}{x^2}$

$$y = \frac{a^3}{a^2}$$

$$\boxed{y = a}$$

$$\therefore x = a, y = a.$$

$\therefore$ The stationary point is (a, a) .

$$r = \frac{d^2f}{dx^2} = \frac{a^3}{x^4} (2x) = \frac{2a^3}{x^3}$$

$$S = \frac{d^2f}{dx dy} = 1$$

$$t = \frac{d^2f}{dy^2} = \frac{a^3}{y^4} (2y) = \frac{2a^3}{y^3}$$

At the point (a, a)

$$r = \frac{2a^3}{a^3} = 2, \quad S = 1, \quad t = \frac{2a^3}{a^3} = 2.$$

$$rt - S^2$$

$$= (2)(2) - (1)^2$$

$$= 4 - 1 = 3 > 0.$$

$$rt - S^2 > 0, \quad r = 2 > 0.$$

$\therefore$ The function has minimum value at the point (a, a) .

$$\begin{aligned} \therefore \text{Minimum value is } f &= (a)(a) + \frac{a^3}{a} + \frac{a^3}{a} \\ &= a^2 + a^2 + a^2 \\ &= \underline{\underline{3a^2}} \end{aligned}$$